

STEP II, 2012, Q3 MS

Question 3

It helps greatly to begin with, to note that if $t = \sqrt{x^2 + 1} + x$, then $\frac{1}{t} = \sqrt{x^2 + 1} - x$. These then give the result $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$, from which we find $\frac{dx}{dt} = \frac{1}{2} + \frac{1}{2}t^{-2}$ and (changing the limits) $x : (0, \infty) \rightarrow t : (1, \infty)$, so that $\int_0^{\infty} f(\sqrt{x^2 + 1} + x) dx = \int_1^{\infty} f(t) \times \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt = \frac{1}{2} \int_1^{\infty} f(x) \left(1 + \frac{1}{x^2}\right) dx$, as required.

For the first integral, $I_1 = \int_0^{\infty} \frac{1}{(\sqrt{x^2 + 1} + x)^2} dx$, we are using $f(x) = \frac{1}{x^2}$ in the result established initially.

$$\text{Then } I_1 = \frac{1}{2} \int_1^{\infty} \left(1 + \frac{1}{x^2}\right) \cdot \frac{1}{x^2} dx = \frac{1}{2} \int_1^{\infty} (x^{-2} + x^{-4}) dx = \frac{1}{2} \left[-\frac{1}{x} - \frac{1}{3x^3} \right]_1^{\infty} = \frac{1}{2} \left(0 + 1 + \frac{1}{3}\right) = \frac{2}{3}.$$

In the case of the second integral, the substitution $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$. Also $\sqrt{1 + x^2} = \sec \theta$ and the required change of limits yields $(0, \frac{1}{2}\pi) \rightarrow (0, \infty)$. We then have

$$\begin{aligned} I_2 &= \int_0^{\frac{1}{2}\pi} \frac{1}{(1 + \sin \theta)^3} d\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{\sec \theta}{\sec \theta + \tan \theta} \right)^3 d\theta \quad [\text{Note the importance of changing to sec and tan}] \\ &= \int_0^{\frac{1}{2}\pi} \frac{\sec \theta}{(\sec \theta + \tan \theta)^3} \cdot \sec^2 \theta d\theta = \int_0^{\infty} \frac{\sqrt{x^2 + 1}}{(\sqrt{x^2 + 1} + x)^3} dx. \end{aligned}$$

We now note, matching this up with the initial result, that we are using $f(t) = \frac{\frac{1}{2} \left(t + \frac{1}{t} \right)}{t^3} = \frac{t^2 + 1}{2t^4}$, so that

$$I_2 = \frac{1}{2} \int_1^{\infty} \left(\frac{t^2 + 1}{t^2} \right) \left(\frac{t^2 + 1}{2t^4} \right) dt = \frac{1}{4} \int_1^{\infty} (t^{-2} + 2t^{-4} + t^{-6}) dt = \frac{1}{4} \left[-\frac{1}{t} - \frac{2}{3t^3} - \frac{1}{5t^5} \right]_1^{\infty} = \frac{1}{4} \left(0 + 1 + \frac{2}{3} + \frac{1}{5} \right) = \frac{7}{15}.$$



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