STEP II, 2012, Q3 MS

Question 3

It helps greatly to begin with, to note that if $t = \sqrt{x^2+1}+x$, then $\frac{1}{t} = \sqrt{x^2+1}-x$. These then give the result $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$, from which we find $\frac{dx}{dt} = \frac{1}{2} + \frac{1}{2}t^{-2}$ and (changing the limits) $x : (0, \infty) \to t : (1, \infty)$, so that $\int_0^\infty f\left(\sqrt{x^2+1}+x\right)\,dx = \int_1^\infty f(t) \times \frac{1}{2}\left(1+\frac{1}{t^2}\right)\,dt = \frac{1}{2}\int_1^\infty f(x)\left(1+\frac{1}{x^2}\right)\,dx$, as required.

For the first integral, $I_1 = \int_0^\infty \frac{1}{\left(\sqrt{x^2+1}+x\right)^2}\,dx$, we are using $f(x) = \frac{1}{x^2}$ in the result established initially.

Then $I_1 = \frac{1}{2}\int_1^\infty \left(1+\frac{1}{x^2}\right).\frac{1}{x^2}\,dx = \frac{1}{2}\int_1^\infty \left(x^{-2}+x^{-4}\right)\,dx = \frac{1}{2}\left[-\frac{1}{x}-\frac{1}{3x^3}\right]_1^\infty = \frac{1}{2}\left(0+1+\frac{1}{3}\right) = \frac{2}{3}$.

In the case of the second integral, the substitution $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$. Also $\sqrt{1+x^2} = \sec\theta$ and the required change of limits yields $\left(0, \frac{1}{2}\pi\right) \to (0, \infty)$. We then have

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,d\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{\sec\theta}{\sec\theta+\tan\theta}\right)^3 d\theta \quad \text{[Note the importance of changing to sec and tan]}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec\theta}{(\sec\theta+\tan\theta)^3}.\sec^2\theta\,d\theta = \int_0^\infty \frac{\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)^3}\,dx.$$

We now note, matching this up with the initial result, that we are using $f(t) = \frac{\frac{1}{2}\left(t+\frac{1}{t}\right)}{t^3} = \frac{t^2+1}{2t^4}$, so that

$$I_2 = \frac{1}{2}\int_1^\infty \left(\frac{t^2+1}{t^2}\right)\left(\frac{t^2+1}{2t^4}\right)dt = \frac{1}{4}\int_1^\infty \left(t^{-2}+2t^{-4}+t^{-6}\right)dt = \frac{1}{4}\left[-\frac{1}{t}-\frac{2}{3t^3}-\frac{1}{5t^5}\right]_1^\infty = \frac{1}{4}\left(0+1+\frac{2}{3}+\frac{1}{5}\right) = \frac{7}{15}.$$