STEP II, 2012, Q2 MS

Question 2

Firstly, $\mathrm{p}(\mathrm{q}(x))$ has degree mn.

(i) $\text{Deg}[\mathrm{p}(x)] = n \Rightarrow \text{Deg}[\mathrm{p}(\mathrm{p}(x))] = n^2$ & $\text{Deg}[\mathrm{p}(\mathrm{p}(\mathrm{p}(x)))] = n^3$.

$\text{Deg}[\text{LHS}] \le \max(n^3, n)$ while RHS is of degree 1. Therefore the LHS is not constant so $n = 1$ and $\mathrm{p}(x)$ is linear. Setting $\mathrm{p}(x) = ax + b \Rightarrow \mathrm{p}(\mathrm{p}(x)) = a(ax + b) + b = a^2x + (a + 1)b$ and

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) = a[a^2x + (a + 1)b] + b = a^3x + (a^2 + a + 1)b.$$

Then $a^3x + (a^2 + a + 1)b - 3ax - 3b + 2x \equiv 0 \Rightarrow (a^3 - 3a + 2)x + (a^2 + a - 2)b \equiv 0$

$$\Rightarrow (a - 1)(a^2 + a - 2)x + (a^2 + a - 2)b \equiv 0$$
$$\Rightarrow (a^2 + a - 2)[(a - 1)x + b] \equiv 0$$
$$\Rightarrow (a + 2)(a - 1)[(a - 1)x + b] \equiv 0$$

We have, then, that $a = -2$ or 1. In either case, b takes any (arbitrary) value and the solutions are thus

$$\mathrm{p}_1(x) = -2x + b \quad \text{and} \quad \mathrm{p}_2(x) = x + b.$$

(ii) $\text{Deg}[\text{RHS}] = 4$ while $\text{Deg}[\text{LHS}] \le \max(n^2, 2n, n)$, so it follows that $n = 2$ and $\mathrm{p}(x)$ is quadratic. Setting $\mathrm{p}(x) = ax^2 + bx + c$, we have

$$2\mathrm{p}(\mathrm{p}(x)) = 2a(ax^2 + bx + c)^2 + 2b(ax^2 + bx + c) + 2c$$
$$= 2a\{a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2\} + 2b(ax^2 + bx + c) + 2c$$

$$3(p(x))^2 = 3[a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2] \text{ and } -4\mathrm{p}(x) = -4ax^2 - 4bx - 4c.$$

Thus, LHS $= (2a^3 + 3a^2)x^4 + (4a^2b + 6ab)x^3 + (2ab^2 + 4a^2c + 2ab + 3b^2 + 6ac - 4a)x^2$
$+ (4abc + 2b^2 + 6bc - 4b)x + (2ac^2 + 2bc + 2c + 3c^2 - 4c)$,

while the RHS $= x^4$.

Equating terms gives

x^4) $\quad 2a^3 + 3a^2 - 1 = 0 \Rightarrow (a + 1)^2(2a - 1) \Rightarrow a = -1$ or $\frac{1}{2}$

x^3) $\quad 2ab(2a + 3) = 0 \Rightarrow b = 0$

x^2) $\quad 2a(2ac + 3c - 2) = 0 \Rightarrow c = 2$ when $a = -1$; i.e. $\mathrm{p}_1(x) = -x^2 + 2$

OR $c = \frac{1}{2}$ when $a = \frac{1}{2}$; i.e. $\mathrm{p}_2(x) = \frac{1}{2}(x^2 + 1)$.

Note that there are two sets of conditions yet to be used, so the results obtained need to be checked (visibly) for consistency:

x^1) $2b(2ac + b + 3c - 2) = 0$ checks and x^0) $\quad c(2ac + 3c - 2) = 0$ checks also.