STEP II, 2012, Q1 MS

Question 1

To be honest, the binomial expansions of $(1 \pm x)^n$, in the cases $n = 1, 2$, are used so frequently within AS- and A-levels that they should be familiar to all candidates taking STEPs. Replacing x by x^k is no great further leap.

The general term in $(1 - x^6)^{-2}$ is easily seen to be $(n + 1)\,x^{6n}$ and the x^{24} term in $(1 - x^6)^{-2}(1 - x^3)^{-1}$ comes from $1.x^{24} + 2x^6.x^{18} + 3x^{12}.x^{12} + 4x^{18}.x^6 + 5x^{24}.1$, so that the coefficient of x^{24} is $1 + 2 + 3 + 4 + 5 = 15$, arising from a sum of triangular numbers. Thus, the coefficient of x^n is

$$\begin{cases} 0 & \text{if } n = 6k + \{1, 2, 4, 5\} \\ \frac{1}{2}(k+1)(k+2) & \text{if } n = 6k + 3 \\ \frac{1}{2}(k+1)(k+2) & \text{if } n = 6k \end{cases}$$

which is most easily described without using n directly, as here.

In (ii), $\text{f}(x) = (1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots)(1 + x^3 + x^6 + x^9 + \ldots)(1 + x + x^2 + x^3 + \ldots)$ and the x^{24} term comes from

$$\begin{aligned}
1.1.5x^{24} &+ 1.x^6.4x^{18} + 1.x^{12}.3x^{12} + 1.x^{18}.2x^6 + 1.x^{24}.1 \\
&+ x^3.x^3.4x^{18} + x^3.x^9.3x^{12} + x^3.x^{15}.2x^6 + x^3.x^{21}.1 \\
&+ x^6.1.4x^{18} + x^6.x^6.3x^{12} + x^6.x^{12}.2x^6 + x^6.x^{18}.1 \\
&+ x^9.x^3.3x^{12} + x^9.x^9.2x^6 + x^9.x^{15}.1 \\
&+ x^{12}.1.3x^{12} + x^{12}.x^6.2x^6 + x^{12}.x^{12}.1 \\
&+ x^{15}.x^3.2x^6 + x^{15}.x^9.1 \\
&+ x^{18}.1.2x^6 + x^{18}.x^6.1 \\
&+ x^{21}.x^3.1 \\
&+ x^{24}.1.1
\end{aligned}$$

giving the coefficient of x^{24} as $15 + 2 \times (10 + 6 + 3 + 1) = 55$.

However, there are lots of ways to go about doing this. For instance ...
Note that, because every non-multiple-of-3 power in bracket 3 is redundant, the x^{24} term comes from considering $\text{f}(x) = (1 - x^6)^{-2}(1 - x^3)^{-2} = (1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots)(1 + 2x^3 + 3x^6 + 4x^9 + \ldots)$.
Again, every non-multiple-of-6 power in *this* 2nd bracket is also redundant, so one might consider only

$$\text{f}(x) = (1 + 3x^6 + 5x^{12} + 7x^{18} + 9x^{24} + \ldots)(1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots)$$

from which the coefficient of x^{24} is simply calculated as $1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55$. This result, in some form or another, gives the way of working out the coefficient of x^{6n} for any non-negative integer n. It is immediately obvious that it is $\sum_{r=0}^{n} (n + 1 - r)(2r + 1)$ which turns out to be the same as $\sum_{r=1}^{n+1} r^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$. The proof of this result could be by induction or direct manipulation of the standard results for Σr and Σr^2.

The coefft. of x^{25} is 55, the same as for x^{24}, since the extra x only arises from replacing 1 by x, x^3 by x^4, etc., in the first bracket's term (at each step of the working) and the coefficients are equal in each case.

In the case when $n = 11$, the coefficient of x^{66} is $12 \times 1 + 11 \times 3 + 10 \times 5 + \ldots + 2 \times 21 + 1 \times 23 = 650$.