STEP II, 2012, Q13 MS

Question 13

Working with the distribution $\mathrm{P_o}(\lambda = k\pi y^2)$, P(no supermarkets) = $\mathrm{e}^{-k\pi y^2}$ and $\mathrm{P}(Y < y) = 1 - \mathrm{e}^{-k\pi y^2}$.

Differentiating w.r.t. y to find the pdf of $Y \Rightarrow$ $\mathrm{f}(y) = 2k\pi\, y\,\mathrm{e}^{-k\pi y^2}$, as given. Then

$\mathrm{E}(Y) = \int_0^\infty 2k\pi\, y^2\,\mathrm{e}^{-k\pi y^2}\,\mathrm{d}y$. Using *Integration by Parts* and writing $2k\pi\, y^2\,\mathrm{e}^{-k\pi y^2}$ as $y\left(2k\pi\, y\,\mathrm{e}^{-k\pi y^2}\right)$

gives $\mathrm{E}(Y) = \left[y\left(\mathrm{e}^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty \mathrm{e}^{-k\pi y^2}\,\mathrm{d}y = 0 + \int_0^\infty \mathrm{e}^{-k\pi y^2}\,\mathrm{d}y$. It is useful (but not essential) to use the

simplifying substitution $x = y\sqrt{2k\pi}$ at this stage to get $\dfrac{1}{\sqrt{2k\pi}}\int_0^\infty \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = \dfrac{1}{\sqrt{2k\pi}}\sqrt{\dfrac{\pi}{2}} = \dfrac{1}{2\sqrt{k}}$ (by the given result, relating to the standard normal distribution's pdf, at the very beginning of the question).

Next, $\mathrm{E}(Y^2) = \int_0^\infty 2k\pi\, y^3\,\mathrm{e}^{-k\pi y^2}\,\mathrm{d}y$, and using *Integration by Parts* and, in a similar way to earlier,

writing $2k\pi\, y^3\,\mathrm{e}^{-k\pi y^2}$ as $y^2\left(2k\pi\, y\,\mathrm{e}^{-k\pi y^2}\right)$, $\mathrm{E}(Y^2) = \left[y^2\left(\mathrm{e}^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty 2y\,\mathrm{e}^{-k\pi y^2}\,\mathrm{d}y$

$= 0 + \dfrac{1}{k\pi}\int_0^\infty 2k\pi\, y\,\mathrm{e}^{-k\pi y^2} = \dfrac{-1}{k\pi}\left[\mathrm{e}^{-k\pi y^2}\right]_0^\infty$ (using a previous result, or by substitution) $= \dfrac{1}{k\pi}$

$\Rightarrow \mathrm{Var}(Y) = \dfrac{1}{k\pi} - \dfrac{1}{4k} = \dfrac{4-\pi}{4k\pi}$, the given answer, as required.