STEP II, 2012, Q12 MS

Question 12

This *can* be broken down into more (four) separate cases, but there is no need to:

P(light on) $= p \times \frac{3}{4} \times \frac{1}{2} + (1-p) \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}(1+2p)$, and then the conditional probability

$$\text{P(Hall} \mid \text{on)} = \frac{\frac{1}{8}(1-p)}{\frac{1}{8}(1+2p)} = \frac{(1-p)}{(1+2p)}.$$

To make progress with this next part of the question, it is important to recognise the underlying binomial distribution, and that each day represents one such (Bernouilli) trial. We are thus dealing with B(7, p_1), where $p_1 = \dfrac{(1-p)}{(1+2p)}$ is the previously given answer.

For the modal value to be 3, we must have P(2) < P(3) < P(4); that is,

$$\binom{7}{2}(p_1)^2(1-p_1)^5 < \binom{7}{3}(p_1)^3(1-p_1)^4 \quad \textit{and} \quad \binom{7}{4}(p_1)^4(1-p_1)^3 < \binom{7}{3}(p_1)^3(1-p_1)^4.$$

Using $p_1 = \dfrac{(1-p)}{(1+2p)}$ gives

$$21\left(\frac{1-p}{1+2p}\right)^2\left(\frac{3p}{1+2p}\right)^5 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow 3(3p) < 5(1-p) \Rightarrow p < \tfrac{5}{14}$$

and

$$35\left(\frac{1-p}{1+2p}\right)^4\left(\frac{3p}{1+2p}\right)^3 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow (1-p) < (3p) \Rightarrow p > \tfrac{1}{4}.$$