STEP II, 2011, Q7 MS

Q7 **(i)** Once you have split each series into sums of powers of λ and μ separately, it becomes clear that you are merely dealing with GPs. Thus $\sum_{r=0}^{n} b_r = \left(1+\lambda+\lambda^2+\ldots+\lambda^n\right)-\left(1+\mu+\mu^2+\ldots+\mu^n\right)$

$$= \frac{\lambda^{n+1}-1}{\lambda-1} - \frac{\mu^{n+1}-1}{\mu-1} = \frac{1}{\sqrt{2}}\left(\lambda^{n+1}-1+\mu^{n+1}-1\right), \text{ since } \lambda-1=\sqrt{2} \text{ and } \mu-1=-\sqrt{2}$$

$$= \frac{1}{\sqrt{2}}a_{n+1}-\sqrt{2} \text{ and, similarly, } \sum_{r=0}^{n} a_r = \frac{\lambda^{n+1}-1}{\sqrt{2}} - \frac{\mu^{n+1}-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}b_{n+1}.$$

(ii) There is no need to be frightened by the appearance of the nested sums here as the 'inner sum' has already been computed: all that is left is to work with the remaining 'outer sum' and deal carefully with the limits: $\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \sum_{m=0}^{2n}\left(\frac{1}{\sqrt{2}}b_{m+1}\right) = \frac{1}{\sqrt{2}}\sum_{m=0}^{2n+1} b_m$ (since $b_0=0$)

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}a_{2n+2}-\sqrt{2}\right) = \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right) = \frac{1}{2}\left(\left[\lambda^{n+1}\right]^2 - 2\left[\lambda\mu\right]^{n+1} + \left[\mu^{n+1}\right]^2\right) \text{ since } \lambda\mu=-1$$

and $n+1$ is even when n is odd $= \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd. However, when n is even, $n+1$ is odd and $\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \frac{1}{2}\left(b_{n+1}\right)^2 - 2$ or $\frac{1}{2}\left(a_{n+1}\right)^2$.

(iii) We already have the result $\left(\sum_{r=0}^{n} a_r\right)^2 = \frac{1}{2}\left(b_{n+1}\right)^2$, so the only new thing is

$\sum_{r=0}^{n} a_{2r+1} = \left(\lambda+\lambda^3+\lambda^5+\ldots+\lambda^{2n+1}\right)+\left(\mu+\mu^3+\mu^5+\ldots+\mu^{2n+1}\right)$, which is still the sum of two GPs, merely with different common ratios, having sum $\frac{\lambda\left(\lambda^{2n+2}-1\right)}{\lambda^2-1}+\frac{\mu\left(\mu^{2n+2}-1\right)}{\mu^2-1}$.

Now $\lambda^2-1 = 3+2\sqrt{2}-1 = 2\left(1+\sqrt{2}\right) = 2\lambda$ and $\mu^2-1 = 3-2\sqrt{2}-1 = 2\left(1-\sqrt{2}\right) = 2\mu$,

so $\sum_{r=0}^{n} a_{2r+1} = \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right) = \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd, and $\frac{1}{2}\left(b_{n+1}\right)^2 - 2$ when n is even.

Thus $\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 0$ when n is odd / $= 2$ when n is even.