STEP II, 2011, Q6 MS

Q6 To begin with, it is essential to realise that the integrand of $I = \int [f'(x)]^2 [f(x)]^n \, dx$ must have its two components split up suitably so that integration by parts can be employed. Thus

$$I = \int [f'(x)] \times \{[f'(x)][f(x)]^n\} dx = f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \int \left([f''(x)] \times \frac{1}{n+1}[f(x)]^{n+1} \right) dx.$$

Now (and not earlier) is the opportune moment to use the given relationship $f''(x) = kf(x)f'(x)$,

so that $I = f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \int \left(kf'(x) \times \frac{1}{n+1}[f(x)]^{n+2} \right) dx$, which is now directly integrable

as $f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \frac{1}{(n+1)(n+3)} \times k[f(x)]^{n+3}$ $(+\,C)$.

(i) For $f(x) = \tan x$, $f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x = kf(x)f'(x)$ with $k = 2$.

Also, differentiating $I = \frac{\sec^2 x \tan^{n+1} x}{n+1} - \frac{2\tan^{n+3} x}{(n+1)(n+3)}$ gives

$$\frac{dI}{dx} = \frac{1}{n+1}\left(\sec^2 x.(n+1)\tan^n x.\sec^2 x + 2\sec x.\sec x \tan x.\tan^{n+1} x\right)$$

$$-\frac{1}{(n+1)(n+3)}\left(2(n+3)\tan^{n+2} x.\sec^2 x\right) = \sec^4 x \tan^n x = (f'(x))^2 \times (f(x))^n$$ as required,

although this could be verified in reverse using integration. Using this result directly in the first given integral is now relatively straightforward:

$$\int \frac{\sin^4 x}{\cos^8 x} dx = \int \sec^4 x \tan^4 x \, dx = \frac{\sec^2 x \tan^5 x}{5} - \frac{2\tan^7 x}{35} + C.$$

(ii) Hopefully, all this differentiating of sec and tan functions may have helped you identify the right sort of area to be searching for ideas with the second of the given integrals.

If $f(x) = \sec x + \tan x$, $f'(x) = \sec x \tan x + \sec^2 x = \sec x(\sec x + \tan x)$

and $f''(x) = \sec^2 x(\sec x + \tan x) + \sec x \tan x(\sec x + \tan x)$

$= \sec x\,(\sec x + \tan x)^2 = kf(x)f'(x)$ with $k = 1$.

Then $\int \sec^2 x(\sec x + \tan x)^6 dx = \int \{\sec x(\sec x + \tan x)\}^2 \times (\sec x + \tan x)^4 dx$

$$= \frac{\sec x(\sec x + \tan x)^6}{5} - \frac{(\sec x + \tan x)^7}{35} + C$$