STEP II, 2011, Q5 MS

Q5 The simplest way to do this is to realise that OA is the bisector of $\angle BOC$, so that A is on the diagonal OA' of parallelogram $OBA'C$ (in fact, since $OB = OC$, it is a rhombus) $\Rightarrow$ b + c = λa for some λ (giving the first part of the result). Also, as BC is perpendicular to OA, $(\mathbf{b} - \mathbf{c}) \,.\, \mathbf{a} = 0$ $\Rightarrow (2\mathbf{b} - \lambda\mathbf{a}) \,.\, \mathbf{a} = 0 \Rightarrow \lambda = 2\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{a} \bullet \mathbf{a}}\right)$.

Similarly (replacing **a** by **b** and **b** by **c** in the above), we have $\mathbf{d} = k\mathbf{b} - \mathbf{c}$ where $k = 2\left(\dfrac{\mathbf{b} \bullet \mathbf{c}}{\mathbf{b} \bullet \mathbf{b}}\right)$ $= 2\left(\dfrac{\mathbf{b} \bullet \lambda\mathbf{a} - \mathbf{b} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) = 2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 2 \Rightarrow \mathbf{d} = \left(2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 2\right)\mathbf{b} - (\lambda\mathbf{a} - \mathbf{b}) = \mu\mathbf{b} - \lambda\mathbf{a}$ where $\mu = 2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 1$ or $4\left(\dfrac{[\mathbf{a} \bullet \mathbf{b}]^2}{[\mathbf{a} \bullet \mathbf{a}][\mathbf{b} \bullet \mathbf{b}]}\right) - 1$.

Now A, B and D are collinear if and only if $\overrightarrow{AD} = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$ is a multiple of $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ $\Leftrightarrow t(\mathbf{b} - \mathbf{a}) = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$ for some $t\ (\neq 0)$.

Comparing coefficients of **a** and **b** then gives $(t =)\ \mu = \lambda + 1$.

In the case when $\lambda = -\tfrac{1}{2}$, $\mu = \tfrac{1}{2}$ and D is the midpoint of AB.

Finally, $\mu = \tfrac{1}{2} \Rightarrow \tfrac{1}{2} = 4\left(\dfrac{[\mathbf{a} \bullet \mathbf{b}]^2}{[\mathbf{a} \bullet \mathbf{a}][\mathbf{b} \bullet \mathbf{b}]}\right) - 1 = 4\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{ab}\right)^2 - 1$, and using the scalar product formula $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{ab}$ gives $\cos\theta = -\sqrt{\dfrac{3}{8}}$. [Note that $\mathbf{a} \,.\, \mathbf{b}$ has the same sign as λ.]