# STEP II, 2011, Q4 MS

**Q4** **(i)** Using $\sin A = \cos(90^\circ - A)$ gives $\theta = 360n \pm (90^\circ - 4\theta)$ – Note that you certainly should be aware of the periodicities of the basic trig. functions $\Rightarrow 5\theta = 360n + 90^\circ$ or $3\theta = 360n + 90^\circ$. These give either $\theta = 72n + 18^\circ \Rightarrow \theta = 18^\circ, 90^\circ, 162^\circ$ or $\theta = 120n + 30^\circ \Rightarrow \theta = 30^\circ, 150^\circ$.

Now using the double-angle formulae for sine (twice) and cosine, we have $c = 2.2sc.(1 - 2s^2)$. We can discount $c = 0$ for $\theta = 18^\circ$, so that $1 = 4s(1 - 2s^2)$ which gives the cubic equation in $s = \sin\theta$, $8s^3 - 4s + 1 = 0 \Rightarrow (2s - 1)(4s^2 + 2s - 1) = 0$. Again, we can discount $c = \frac{1}{2}$ for $\theta = 18^\circ$) which leaves us with $\sin 18^\circ$ the positive root (as $18^\circ$ is acute) from the two possible solutions of this quadratic; namely, $\sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$.

**(ii)** Using the double-angle formula for sine, we have $4s^2 + 1 = 16s^2(1 - s^2) \Rightarrow 0 = 16s^4 - 12s^2 + 1$ $\Rightarrow s^2 = \dfrac{12 \pm \sqrt{80}}{32} = \dfrac{3 \pm \sqrt{5}}{8}$. At first, this may look like a problem, but bear in mind that we want it to be a perfect square. Proceeding with this in mind, $s^2 = \dfrac{6 \pm 2\sqrt{5}}{16} = \left(\dfrac{\sqrt{5} \pm 1}{4}\right)^2$ so that we have the four answers, $\sin x = \pm\left(\dfrac{\sqrt{5} \pm 1}{4}\right)$.

**(iii)** To make the connection between this part and the previous one requires nothing more than division by 4 to get $\sin^2 x + \frac{1}{4} = \sin^2 2x$, and the solution $x = 3\alpha = 18^\circ$, $5\alpha = 30^\circ \Rightarrow \alpha = 6^\circ$ immediately presents itself from part (ii). However, in order to **deduce** a second solution (noting that $\alpha = 45^\circ$ is easily seen to satisfy the given equation), it is important to be prepared to be a bit flexible and use your imagination. The other possible angles that are "related" to $18^\circ$ and might satisfy (ii)'s equation, can be looked-for, provided that $\sin 5\alpha = \pm\frac{1}{2}$ (and there are many possibilities here also). A little searching and/or thought reveals
$\sin x = -\left(\dfrac{\sqrt{5} - 1}{4}\right) \Rightarrow 3\alpha = 180^\circ + 18^\circ = 198^\circ$ also works, since $5\alpha = 330^\circ$ has $\sin 5\alpha = -\frac{1}{2}$, and the second acute answer is $\alpha = 66^\circ$.