STEP II, 2011, Q3 MS

Q3 This question is all about increasing functions and what can be deduced from them. It involves inequalities, which are never popular creatures even amongst STEP candidates. Fortunately, you are led fairly gently by the hand into what to do, at least to begin with.

(i) (a) $f'(x) = \cos x - \{x.{-\sin x} + \cos x\} = x \sin x \geq 0$ for $x \in [0, \tfrac{1}{2}\pi]$, and since $f(0) = 0$ it follows that $f(x) = \sin x - x\cos x \geq 0$ for $0 \leq x \leq \tfrac{1}{2}\pi$.

(i) (b) A key observation here is that the "1" is simply a disguise for $\frac{d}{dx}(x)$, so you are actually being given that $\frac{d}{dx}(\arcsin x) \geq \frac{d}{dx}(x)$ in the given interval; in other words, that $f(x) = \arcsin x - x$ is an increasing function. Since $f(0) = 0$ and f increasing, $f(x) = \arcsin x - x \geq 0$ for $0 \leq x < 1$, and the required result follows.

(i) (c) Writing $g(x) = \dfrac{x}{\sin x} \Rightarrow g'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} > 0$ for $0 < x < \tfrac{1}{2}\pi$ using (a)'s result. Now, it may help to write $u = \arcsin x$, just so that it looks simpler to deal with here. Then $u \geq x$ by (b)'s result $\Rightarrow g(u) \geq g(x)$ since $g'(x) \geq 0$ and the required result again follows.

(ii) There is a bit more work to be done here, but essentially the idea is the same as that in part (i), only the direction of the inequality seems to be reversed, so care must be taken. An added difficulty also arises in that we find that we must show that $f' \geq 0$ by showing that it is increasing from zero. So $g(x) = \dfrac{\tan x}{x}$, $g'(x) = \dfrac{x\sec^2 x - \tan x}{x^2} = \dfrac{2x - \sin 2x}{2x^2\cos^2 x}$. Examining $f(x) = 2x - \sin 2x$ (since the denominator is clearly positive in the required interval): $f(0) = 0$ and $f'(x) = 2 - 2\cos 2x \geq 0$ for $0 < x < \tfrac{1}{2}\pi \Rightarrow f \geq 0 \Rightarrow g'(x) \geq 0 \Rightarrow$ g increasing. Mimicking the conclusion of (i) (c), the reader should now be able to complete the solution.