STEP II, 2011 Q3

3 In this question, you may assume without proof that any function f for which $f'(x) \geqslant 0$ is *increasing*; that is, $f(x_2) \geqslant f(x_1)$ if $x_2 \geqslant x_1$.

(i) **(a)** Let $f(x) = \sin x - x\cos x$. Show that $f(x)$ is increasing for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and deduce that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(b) Given that $\dfrac{d}{dx}(\arcsin x) \geqslant 1$ for $0 \leqslant x < 1$, show that

$$\arcsin x \geqslant x \qquad (0 \leqslant x < 1).$$

(c) Let $g(x) = x \operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)\, x^{-1} \geqslant x \operatorname{cosec} x \qquad (0 < x < 1).$$

(ii) Given that $\dfrac{d}{dx}(\arctan x) \leqslant 1$ for $x \geqslant 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geqslant x^2 \qquad (0 < x < \tfrac{1}{2}\pi).$$