STEP II, 2011, Q2 MS

Q2 The required list of perfect cubes is 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, though there were no marks for noting them.

(i) In this question, it is clearly important to be able to factorise the sum of two cubes. So, in this first instance $x + y = k$, $(x+y)(x^2 - xy + y^2) = kz^3 \;\Rightarrow\; x^2 - (k-x)x + (k-x)^2 - z^3 = 0$, which gives the required result upon rearrangement. One could either treat this as a quadratic in x and deal with its discriminant or go ahead directly to show that $\frac{4z^3 - k^2}{3} = (y-x)^2 \ge 0$ which immediately gives that $\frac{4z^3 - k^2}{3}$ is a perfect square and also that $z^3 \ge \frac{1}{4}k^2$; and the other half of the required inequality comes either from $z^3 = k^2 - 3xy < k^2$ (since $x, y > 0$) or from noting that the smaller root of the quadratic in x is positive. Substituting $k = 20$ into the given inequality then yields $100 \le z^3 < 400 \;\Rightarrow\; z = 5, 6, 7$; and the only value of z in this list for which $\frac{4z^3 - k^2}{3}$ is a perfect square is $z = 7$, which then yields the solution $(x, y, z) = (1, 19, 7)$. Although not a part of the question, we can now express 20 as a sum of two rational cubes in the following way:

$$20 = \left(\frac{1}{7}\right)^3 + \left(\frac{19}{7}\right)^3.$$

(ii) Although this second part of the question can be done in other ways, the intention is clearly that a similar methodology to (i)'s can be employed. Starting from

$$x + y = z^2,\; (x+y)(x^2 - xy + y^2) = kz.z^2 \;\Rightarrow\; x^2 - (z^2 - x)x + (z^2 - x)^2 - kz = 0$$

we find that $\frac{4kz - z^4}{3}$ is a perfect square, and also that $k < z^3 \le 4k$. With $k = 19$, $19 < z^3 \le 76$ $\Rightarrow\; z = 3$ or 4. This time, each of these values of z gives $\frac{z(76 - z^3)}{3}$ a perfect square, yielding the two solutions $(x, y, z) = (1, 8, 3)$ and $(6, 10, 4)$. Thus we have two ways to represent 19 as a sum of two rational cubes: $19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$ and $\left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3$.

Purely as an aside, interested students may like to explore other possibilities for $x^3 + y^3 = kz^3$. One that never made it into the question was

$$x + y = kz,\; (x+y)(x^2 - xy + y^2) = kz.z^2 \;\Rightarrow\; x^2 - (kz - x)x + (kz - x)^2 - z^2 = 0$$

$\Rightarrow\; 3x^2 - 3kzx + z^2(k^2 - 1) = 0$. Then $x = \frac{3kz \pm \sqrt{9k^2z^2 - 12z^2(k^2 - 1)}}{6} = \frac{1}{6}z\left\{3k \pm \sqrt{12 - 3k^2}\right\}$, requiring $12 - 3k^2 \ge 0$ i.e. $k^2 \le 4 \;\Rightarrow\; k = 1$ or 2.

When $k = 1$: $x^3 + y^3 = z^3$ has NO solutions by *Fermat's Last Theorem*;

and when $k = 2$: $x^3 + y^3 = 2z^3$ has (trivially) infinitely many solutions $x = y = z$.