STEP II, 2011, Q13 MS

Q13 Firstly, *skewness* is a measure of a distribution's **lack** of symmetry.

(i) For the next part, you should understand how the "expectation" function behaves.

$$E\left[(X-\mu)^3\right] = E\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right] = E\left[X^3\right] - 3\mu E\left[X^2\right] + 3\mu^2 E[X] - \mu^3$$

$$= E\left[X^3\right] - 3\mu\left(\sigma^2 + \mu^2\right) + 3\mu^2.\mu - \mu^3 \text{ using } E[X] = \mu \text{ and } E[X^2] = \sigma^2 + \mu^2$$

$$= E\left[X^3\right] - 3\mu\sigma^2 - \mu^3, \text{ as required.}$$

For a given distribution, this next bit of work is very routine indeed.

$E[X] = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3} = \mu$; $E\left[X^2\right] = \int_0^1 2x^3\,dx = \left[\frac{1}{2}x^4\right]_0^1 = \frac{1}{2} \Rightarrow \sigma^2 = \frac{1}{18}$; and

$E\left[X^3\right] = \int_0^1 2x^4\,dx = \left[\frac{2}{5}x^5\right]_0^1 = \frac{2}{5}$; all of which then lead to $\gamma = \dfrac{\frac{2}{5} - 3.\frac{2}{3}.\frac{1}{18} - \frac{8}{27}}{\frac{1}{18\sqrt{18}}} = -\dfrac{2\sqrt{2}}{5}$ when substituted into the given (previously deduced) formula.

(ii) Here, $F(x) = \int_0^x 2x\,dx = x^2 \ (0 \le x \le 1) \Rightarrow F^{-1}(x) = \sqrt{x} \quad (0 \le x \le 1)$

$$\Rightarrow D = \frac{F^{-1}\left(\frac{9}{10}\right) - 2F^{-1}\left(\frac{1}{2}\right) + F^{-1}\left(\frac{1}{10}\right)}{F^{-1}\left(\frac{9}{10}\right) - F^{-1}\left(\frac{1}{10}\right)} = \frac{\frac{3}{\sqrt{10}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}} = \frac{3 - 2\sqrt{5} + 1}{3 - 1} = \frac{4 - 2\sqrt{5}}{2} = 2 - \sqrt{5}.$$

M is given by $\int_0^M 2x\,dx = \frac{1}{2} \Rightarrow M^2 = \frac{1}{2} \Rightarrow M = \frac{1}{\sqrt{2}}$ (OR by $M = F^{-1}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$).

And $P = \dfrac{3\left(\frac{2}{3} - \frac{1}{\sqrt{2}}\right)}{\frac{1}{3\sqrt{2}}} = 6\sqrt{2} - 9$.

In order to establish the given inequality "chain", we must show that $D > P$ and $P > \gamma$ (there is no point in proving that $D > \gamma$). One could reason this through by considering approximants to $\sqrt{2}$ and $\sqrt{5}$, but care must be taken not to introduce fallacious "roundings" which don't support the direction of the inequality under consideration. The alternative is to establish a set of equivalent numerical statements; for example, to show that $D > P$ …

$$2 - \sqrt{5} > 6\sqrt{2} - 9 \Leftrightarrow 11 - \sqrt{5} > 6\sqrt{2}$$

$$\Leftrightarrow 121 + 5 - 22\sqrt{5} > 72 \text{ (after squaring, since both sides are positive)}$$

$$\Leftrightarrow 54 > 22\sqrt{5} \text{ or } 27 > 11\sqrt{5} \Leftrightarrow 729 > 605 \text{ (again, squaring positive terms)}$$

and this final result clearly *IS* true, so the desired inequality is established.