STEP II, 2011, Q12 MS

Q12 It is important in these sorts of contrived games to read the rules properly: in this case, you must ensure that you are clear what is meant by 'match', 'game' and 'point'. Then, a careful listing of cases is all that is required.

(i) P(re-match) = P(XYX) + P(YXY) = $p(1-p)^2 + (1-p)^3 = (1-p)^2$.

P(Y wins directly) = P(YY) + P(XYY) = $(1-p)p + p(1-p)p = p(1-p)(1+p)$ or $p(1-p^2)$.

Thus, P(Y wins) = $w = p(1-p^2) + w(1-p)^2$, and you will note the way starting the match again leads to a recurrent way of describing Y's chances of winning. Re-arranging this then gives

$$w = \frac{p(1-p^2)}{1-(1-p)^2} = \frac{p(1-p^2)}{(1-(1-p))(1+(1-p))} = \frac{p(1-p^2)}{p(2-p)} = \frac{1-p^2}{2-p} \quad \text{for } p \neq 0.$$

Next, $w - \frac{1}{2} = \frac{2(1-p^2)-(2-p)}{2(2-p)} = \frac{p(1-2p)}{2(2-p)}$, and since $2-p>0$, $w-\frac{1}{2}$ has the same sign as $1-2p$ and hence as $\frac{1}{2}-p$. Hence, $w > \frac{1}{2}$ if $p < \frac{1}{2}$ and $w < \frac{1}{2}$ if $p > \frac{1}{2}$.

To be fair at this point, the final demand of part (i) ended up being rather less demanding than was originally intended, as the answer is either "Yes" or "No" … though you would of course, be expected to support your decision; no marks are given for being a lucky guesser! The following calculus approach is thus slightly unnecessary, as one can simply provide an example to show that w can decrease with p. The following, more detailed analysis had been intended.

$$\frac{dw}{dp} = \frac{(2-p)(-2p)-(1-p^2)(-1)}{(2-p)^2} = \frac{1}{(2-p)^2}(p^2-4p+1) = \frac{1}{(2-p)^2}([2-p]^2-3).$$

Then $\frac{dw}{dp} > 0$ for $0 < p < 2-\sqrt{3}$ and $\frac{dw}{dp} < 0$ for $2-\sqrt{3} < p \leq 1$.

For a fair game, Y's expectation should be 0. Thus, using E(gain) = $\sum g_i \times \mathrm{P}(g_i)$, where g_i is the "gain function" for Y, with $w = \frac{5}{12}$ when $p = \frac{2}{3}$, we have $0 = (k)\times\frac{5}{12} + (-1)\times\frac{7}{12} \Rightarrow k = 1.4$.

When $p = 0$, the results run YXY ... re-match ... YXY ... re-match ... and the match never ends.