STEP II, 2011, Q10 MS

Q10 The maximum height of a projectile is when $\dot{y} = u\sin\theta - gt = 0 \Rightarrow t = \dfrac{u\sin\theta}{g}$. Substituting this into $y = ut\sin\theta - \frac{1}{2}gt^2 \Rightarrow H = \dfrac{u^2\sin^2\theta}{2g}$ (although some people learn it to quote).

When the string goes taut, its length l is given by $l = \frac{1}{2}H = \dfrac{u^2\sin^2\theta}{4g}$. But l is also given from the y-component of P's displacement as $l = ut\sin\theta - \frac{1}{2}gt^2$, which gives the quadratic equation $gt^2 - (2u\sin\theta)t + H = 0$ in t. Solving by the quadratic formula, $t = \dfrac{2u\sin\theta \pm \sqrt{4u^2\sin^2\theta - 4gH}}{2g}$ $= \dfrac{2\sqrt{2gH} \pm \sqrt{8gH - 4gH}}{2g} = \dfrac{1}{g}\left(\sqrt{2gH} \pm \sqrt{gH}\right) = \sqrt{\dfrac{H}{g}}\left(\sqrt{2} - 1\right)$, where we take the smaller of the two roots since we want the first time when an unimpeded P is at this height.

At this time, P's vertical velocity is $v = \dot{y} = u\sin\theta - g\sqrt{\dfrac{H}{g}}\left(\sqrt{2} - 1\right) = \sqrt{2gH} - \sqrt{gH}\left(\sqrt{2} - 1\right)$ $= \sqrt{gH}$ or $\dfrac{u\sin\theta}{\sqrt{2}}$. Thus, the common speed of P/R after the string goes taut, by *CLM*, is $\frac{1}{2}\sqrt{gH}$ or $\dfrac{u\sin\theta}{2\sqrt{2}}$.

When the string goes slack, we must consider the projectile motion of R, which has initial velocity components $u\cos\theta \rightarrow$ and $\dfrac{u\sin\theta}{2\sqrt{2}} \uparrow$. [Note that both P and R move in this way, so P no longer interferes with R's motion.] R's vertical displacement is zero when $y_R = \dfrac{u\sin\theta}{2\sqrt{2}}t - \dfrac{g}{2}t^2 = 0$ $(t \neq 0)$ $\Rightarrow t = \dfrac{u\sin\theta}{g\sqrt{2}}$ (and this is the extra time after the string has gone slack). The total distance travelled by R is thus $D = x_1 + x_2$, where $x_1 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}\left(\sqrt{2} - 1\right)$ and $x_2 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}$ $= \dfrac{u^2\sin\theta\cos\theta}{g}$.

Finally, setting $D = H \Rightarrow \tan\theta = 2$.