STEP II, 2010, Q2

2 Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x\,.$$

Find and prove a similar result for $\sin 3x$ in terms of $\sin x$.

(i) Let

$$\mathrm{I}(\alpha) = \int_0^{\alpha} (7\sin x - 8\sin^3 x)\mathrm{d}x\,.$$

Show that

$$\mathrm{I}(\alpha) = -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}\,,$$

where $c = \cos\alpha$. Write down one value of c for which $\mathrm{I}(\alpha) = 0$.

(ii) Useless Eustace believes that

$$\int \sin^n x\,\mathrm{d}x = \frac{\sin^{n+1} x}{n+1}$$

for $n = 1,\ 2,\ 3, \ldots$. Show that Eustace would obtain the correct value of $\mathrm{I}(\beta)$, where $\cos\beta = -\frac{1}{6}$.

Find all values of α for which he would obtain the correct value of $\mathrm{I}(\alpha)$.