STEP II, 2009, Q7 MS

7 It is easy to saunter into this question's opening without pausing momentarily to wonder if one is going about it in the best way. Whilst many can cope with differentiating a "triple"-product with ease, many others can't. However, even for interests' sake, one might stop to consider a general approach to such matters. Differentiating $y = pqr$ (all implicitly functions of x) as, initially, $p(qr)$ and applying the product-rule twice, one obtains $y' = pq\,r' + p\,q'r + p'qr$, and this can be used here with $p = (x - a)^n$, $q = e^{bx}$ and $r = \sqrt{1+x^2}$ without the need for a lot of the mess (and subsequent mistakes) that was (were) made by so many candidates. Here, $y = (x - a)^n\, e^{bx}\sqrt{1+x^2}$ gives

$$\frac{dy}{dx} = (x-a)^n\, e^{bx}\frac{x}{\sqrt{1+x^2}} + (x-a)^n\, b\, e^{bx}\sqrt{1+x^2} + n(x-a)^{n-1}\, e^{bx}\sqrt{1+x^2}$$

Factorising out the given terms $\Rightarrow \dfrac{(x-a)^{n-1}\, e^{bx}}{\sqrt{1+x^2}}\left\{x(x-a)+b(x-a)(1+x^2)+n(1+x^2)\right\}$, and we are only required to note that the term in the brackets is, indeed, a cubic; though it may prove helpful later on to simplify it by multiplying out and collecting up terms, to get

$$q(x) = bx^3 + (n + 1 - ab)x^2 + (b - a)x + (n - ab).$$

(i) The first integral, $I_1 = \int \dfrac{(x-4)^{14}\, e^{4x}}{\sqrt{1+x^2}}\left(4x^3 - 1\right) dx$, might reasonably be expected to be a very straightforward application of the general result, and so it proves to be. With $n = 15$, and taking $a = b = 4$, so that $q(x) = 4x^3 - 1$ (which really should be checked explicitly), we find

$$I_1 = (x - 4)^{15}\, e^{4x}\sqrt{1+x^2} \quad (+\, C).$$

(ii) This second integral, $I_2 = \int \dfrac{(x-1)^{21}\, e^{12x}}{\sqrt{1+x^2}}\left(12x^4 - x^2 - 11\right) dx$, is clearly not so straightforward, since the bracketed term is now quartic. Of the many things one *might* try, however, surely the simplest is to try to factor out a linear term, the obvious candidate being $(x - 1)$.

Finding that $12x^4 - x^2 - 11 \equiv (x - 1)(12x^3 + 12x^2 + 11x + 11)$, we now try $n = 23$, $a = 1$, $b = 12$ to obtain $q(x) = 12x^3 + 12x^2 + 11x + 11$ and $I_2 = (x - 1)^{23}\, e^{12x}\sqrt{1+x^2}$ $(+\, C)$.

(iii) The final integral, $I_3 = \int \frac{(x-2)^6 \, e^{4x}}{\sqrt{1+x^2}} \left(4x^4 + x^3 - 2\right) \, dx$, is clearly intended to be even less simple than its predecessor. However, you might now suspect that "the next case up" is in there somewhere. So, if you try $n = 8$, $a = 2$, $b = 4$, which gives

$$\frac{dy_8}{dx} = \frac{(x-2)^7 \, e^{4x}}{\sqrt{1+x^2}} \left\{4x^3 + x^2 + 2x\right\} = \frac{(x-2)^6 \, e^{4x}}{\sqrt{1+x^2}} \left\{4x^4 - 7x^3 - 4x\right\},$$

as well as the obvious target $n = 7$, $a = 2$, $b = 4$, which yields

$$\frac{dy_7}{dx} = \frac{(x-2)^6 \, e^{4x}}{\sqrt{1+x^2}} \left\{4x^3 + 2x - 1\right\},$$

It may now be clear that ***both*** are involved. Indeed,

$$I_3 = \int \left(\frac{dy_8}{dx} + 2\frac{dy_7}{dx}\right) dx = y_8 + 2\, y_7 = x(x-2)^7 \, e^{4x} \sqrt{1+x^2} \quad (+\, C).$$
