STEP II, 2009, Q6 MS

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6 If you don't know about the *Fibonacci Numbers* by now, then … shame on you! Nevertheless, the first couple of marks for writing down the next few terms must count as among the easiest on the paper. ($F_1 = 1$, $F_2 = 1$), $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$ and $F_{10} = 55$.

(i) If you're careful, the next section isn't particularly difficult either. Using the recurrence relation gives $\dfrac{1}{F_i} = \dfrac{1}{F_{i-1} + F_{i-2}} > \dfrac{1}{2F_{i-1}}$ since $F_{i-2} < F_{i-1}$ for $i \ge 4$. Splitting off the first few terms then leads to $S = \displaystyle\sum_{i=1}^{n} \frac{1}{F_i} > \frac{1}{F_1} + \frac{1}{F_2}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right)$ or $\dfrac{1}{F_1} + \dfrac{1}{F_2} + \dfrac{1}{F_3}\left(1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots..\right)$, where the long bracket at the end is the sum-to-infinity of a GP. These give, respectively, $S > 1 + 1 \times 2 = 3$ or $1 + 1 + \frac{1}{2} \times 2 = 3$. A simpler approach could involve nothing more complicated than adding the terms until a sum greater than 3 is reached, which happens when you reach F_5.

A similar approach yields $\dfrac{1}{F_i} < \dfrac{1}{2}\left(\dfrac{1}{F_{i-2}}\right)$ for $i \ge 3$ and splitting off the first few terms, this time separating the odd- and even-numbered terms, gives

$$S = \sum_{i=1}^{n} \frac{1}{F_i} = \frac{1}{F_1} + \frac{1}{F_2} + \left(\frac{1}{F_3} + \frac{1}{F_5} + \ldots..\right) + \left(\frac{1}{F_4} + \frac{1}{F_6} + \ldots..\right)$$

$$< 1 + 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right)$$

$$= 1 + 1 + \frac{1}{2} \times 2 + \frac{1}{3} \times 2 = 3\tfrac{2}{3}.$$

(ii) To show that $S > 3.2$, we simply apply the same approaches as before, but taking more terms initially before summing our GP (or stopping at F_7 in the "simpler approach" mentioned previously). Something like

$$S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right) = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 = 3\tfrac{7}{30} > 3\tfrac{6}{30} = 3.2$$

does the job pretty readily. Then, to show that $S < 3\frac{1}{2}$, a similar argument to those you have been directed towards by the question, works well with little extra thought required:

$$S < 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right) + \frac{1}{8}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots..\right)$$

$$- 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2 = 3\tfrac{29}{60} < 3\tfrac{1}{2}.$$

Returning to the initial argument, $F_i < 2\,F_{i-1}$ or $\frac{1}{F_i} > \frac{1}{2}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 4$, we can extend this to $F_i > \frac{3}{2}\,F_{i-1}$ or $\frac{1}{F_i} < \frac{2}{3}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 5$, $F_i < \frac{5}{3}\,F_{i-1}$ or $\frac{1}{F_i} > \frac{3}{5}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 6$, etc., simply by using the defining recurrence relation for the *Fibonacci Numbers*, leading to the general results

$$F_n > \left(\frac{F_{2k}}{F_{2k-1}}\right) F_{n-1} \text{ or } \frac{1}{F_n} < \left(\frac{F_{2k-1}}{F_{2k}}\right)\frac{1}{F_{n-1}} \text{ for } n \ge 2k+1$$

and

$$F_n < \left(\frac{F_{2k+1}}{F_{2k}}\right) F_{n-1} \text{ or } \frac{1}{F_n} > \left(\frac{F_{2k}}{F_{2k+1}}\right)\frac{1}{F_{n-1}} \text{ for } n \ge 2k+2.$$

Since the terms $\frac{F_n}{F_{n-1}} \to \phi = \frac{\sqrt{5}+1}{2}$, the golden ratio, (being the positive root of the quadratic equation $x^2 = x + 1$, we can deduce the approximation $S \approx \sum_{i=1}^{n}\frac{1}{F_i} + \frac{1}{F_{n+1}}\phi^2$ since the geometric progression $1 + \frac{1}{\phi} + \frac{1}{\phi^2} + \ldots = \frac{1}{1-\frac{1}{\phi}} = \frac{\phi}{\phi - 1} = \frac{\phi}{\frac{1}{\phi}} = \phi^2$. Taking $n = 9$, (i.e. just using the first 10 *Fibonacci Numbers* which you were led to write down at the start),

$$S \approx \sum_{i=1}^{9}\frac{1}{F_i} + \frac{1}{F_{10}}\phi^2 = \frac{614893}{185640} + \frac{1}{55} \times \frac{\sqrt{5}+3}{2} \approx 3.359\,89,$$

which is correct to 5 d.p. For further information on this number, try looking up the '*Reciprocal Fibonacci constant*' on Wikipedia, for instance.
