STEP II, 2009, Q5 MS

5 The very first bit is not just a giveaway mark, but rather a helpful indicator of the kind of result or technique that may be used in this question: $\left(\sqrt{x-1}+1\right)^2 = x+2\sqrt{x-1}$; but pay attention to what happens here. Most particularly, the fact that $\left(\sqrt{x-1}+1\right)^2 = x+2\sqrt{x-1}$ does NOT necessarily mean that $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ since positive numbers have *two* square-roots! Recall that $\sqrt{x^2} = |x|$ and not just x. Notice that, during the course of this question, the range of values under consideration switches from (5, 10) to $\left(\frac{5}{4}, 10\right)$, and one doesn't need to be particularly suspicious to wonder why this is so. A modicum of investigation at the outset seems warranted here, as to when things change sign.

(i) So … while $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ seems a perfectly acceptable thing to write, since $x \geq 1$ is a necessary condition in order to be able to take square-roots at all here (for real numbers), simply writing down that $\sqrt{x-2\sqrt{x-1}} = +\left(\sqrt{x-1}-1\right)$ *may* cause a problem. A tiny amount of exploration shows that $\sqrt{x-1}-1$ changes from negative to positive around $x = 2$. Hence, in part (i), we can ignore any negative considerations and plough ahead: $I = \int_5^{10} 2\,\mathrm{d}x = \left[2x\right]_5^{10} = 10$.

(ii) Here in (ii), however, you should realise that the area requested is the sum of two portions, one of which lies below the x-axis, and would thus contribute negatively to the total if you failed to take this into account. Thus,

$$\text{Area} = \int_{1.25}^{2} \frac{1-\sqrt{x-1}}{\sqrt{x-1}}\,\mathrm{d}x + \int_{2}^{10} \frac{\sqrt{x-1}-1}{\sqrt{x-1}}\,\mathrm{d}x = \int_{1.25}^{2}\left[(x-1)^{-\frac{1}{2}}-1\right]\mathrm{d}x + \int_{2}^{10}\left[1-(x-1)^{-\frac{1}{2}}\right]\mathrm{d}x$$

$$= \left[2\sqrt{x-1}-x\right]_{1.25}^{2} + \left[x-2\sqrt{x-1}\right]_{2}^{10} = 4\tfrac{1}{4}.$$

(iii) Now $\left(\sqrt{x+1}-1\right)^2 = x+2-2\sqrt{x+1}\ \ \forall\, x \geq 0$ so we have no cause for concern here. Then

$$I = \int_{x=1.25}^{10} \frac{1+\sqrt{x-1}+\sqrt{x+1}-1}{\sqrt{x-1}\sqrt{x+1}}\,\mathrm{d}x = \int_{x=1.25}^{10}\left((x+1)^{-\frac{1}{2}}+(x-1)^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= \left[2\sqrt{x+1}+2\sqrt{x-1}\right]_{1.25}^{10} = 2\left(\sqrt{11}+1\right)$$