# STEP II, 2009, Q4 MS

**4**(i) Writing $p(x) - 1 = q(x).(x-1)^5$, where $q(x)$ is a quartic polynomial, immediately gives $p(1) = 1$.

(ii) Diff$^g$. using the product and chain rules leads to

$$p'(x) = q(x).5(x-1)^4 + q'(x).(x-1)^5 = (x-1)^4.\{5\,q(x) + (x-1)\,q'(x)\},$$

so that $p'(x)$ is divisible by $(x-1)^4$ .

(iii) Similarly, we have that $p'(x)$ is divisible by $(x+1)^4$ and $p(-1) = -1$ .
Thus $p'(x)$ is divisible by $(x+1)^4.(x-1)^4 = (x^2-1)^4$ . However, $p'(x)$ *is* a polynomial of degree eight, hence $p'(x) = k(x^2-1)^4$ for some constant $k$ . That is, $p'(x) = k\left(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right)$ .
Integrating term by term then gives $p(x) = k\left(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) + C$, and use of both $p(1) = 1$ and $p(-1) = -1$ help to find $k$ and $C$; namely, $k = \dfrac{315}{128}$ and $C = 0$.