

## STEP II, 2009, Q3 MS

3 Using the “addition” formula for  $\tan(A - B)$ ,

$$\begin{aligned} \text{LHS} &= \tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{1 - \tan \frac{\pi}{2}}{1 + \tan \frac{\pi}{2}} = \frac{\cos \frac{\pi}{2} - \sin \frac{\pi}{2}}{\cos \frac{\pi}{2} + \sin \frac{\pi}{2}} = \frac{\cos \frac{\pi}{2} - \sin \frac{\pi}{2}}{\cos \frac{\pi}{2} + \sin \frac{\pi}{2}} \times \frac{\cos \frac{\pi}{2} - \sin \frac{\pi}{2}}{\cos \frac{\pi}{2} - \sin \frac{\pi}{2}} \\ &= \frac{1 - 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2}}{\cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2}} \quad (\text{since } c^2 + s^2 = 1) = \frac{1 - \sin x}{\cos x} = \sec x - \tan x = \text{RHS}. \end{aligned}$$

Alternatively, one could use the “ $t = \tan(\frac{1}{2}$ -angle)” formulae to show that

$$\text{RHS} = \frac{1-t^2}{1+t^2} - \frac{2t}{1-t^2} = \frac{(1-t)^2}{(1-t)(1+t)} = \frac{1-t}{1+t} = \frac{1 - \tan \frac{\pi}{2}}{1 + \tan \frac{\pi}{2}} = \text{LHS}.$$

(i) Setting  $x = \frac{\pi}{4}$  in (\*)  $\Rightarrow \tan \frac{\pi}{4} = \sqrt{2} - 1$ . Then, using the addition formula for  $\tan(A + B)$  with

$$A = \frac{\pi}{3} \text{ and } B = \frac{\pi}{8}, \text{ we have } \tan \frac{11\pi}{24} = \tan\left(\frac{\pi}{3} + \frac{\pi}{8}\right) = \frac{\sqrt{3} + \sqrt{2} - 1}{1 - \sqrt{3}(\sqrt{2} - 1)} = \frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1}, \text{ as required.}$$

(ii) Now, in the “spirit” of maths, one might reasonably expect that one should take the given expression, rationalise the denominator (twice) and derive the given answer, along the lines ...

$$\frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} = \frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} \times \frac{1 + \sqrt{3} + \sqrt{6}}{1 + \sqrt{3} + \sqrt{6}} = \frac{1 + 2\sqrt{2} + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

However, with a given answer, it is perfectly legitimate merely to multiply across and verify that

$$(\sqrt{3} - \sqrt{6} + 1)(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}) = \sqrt{3} + \sqrt{2} - 1.$$

(iii) Having got this far, the end is really very clearly signposted. Setting  $x = \frac{11\pi}{24}$  in (\*) gives

$$\begin{aligned} \tan \frac{\pi}{48} &= \sec \frac{11\pi}{24} - \tan \frac{11\pi}{24} = \sqrt{1+t^2} - t \\ &= \sqrt{1 + \left[4 + 2 + 3 + 6 + 4\sqrt{2} + 4\sqrt{3} + 4\sqrt{6} + 2\sqrt{6} + 2\sqrt{12} + 2\sqrt{18}\right]} - (2 + \sqrt{2} + \sqrt{3} + \sqrt{6}) \\ &= \sqrt{15 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}} - (2 + \sqrt{2} + \sqrt{3} + \sqrt{6}) \end{aligned}$$



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