STEP II, 2009, Q3 MS

3 Using the "addition" formula for $\tan(A - B)$,

$$\text{LHS} = \tan\left(\tfrac{\pi}{4} - \tfrac{x}{2}\right) = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} = \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} = \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \times \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}$$

$$= \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \text{ (since } c^2 + s^2 = 1) = \frac{1 - \sin x}{\cos x} = \sec x - \tan x = \text{RHS.}$$

Alternatively, one could use the "$t = \tan(\frac{1}{2}\text{-angle})$" formulae to show that

$$\text{RHS} = \frac{1 - t^2}{1 + t^2} - \frac{2t}{1 - t^2} = \frac{(1 - t)^2}{(1 - t)(1 + t)} = \frac{1 - t}{1 + t} = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} = \text{LHS.}$$

(i) Setting $x = \frac{\pi}{4}$ in (*) $\Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1$. Then, using the addition formula for $\tan(A + B)$ with $A = \frac{\pi}{3}$ and $B = \frac{\pi}{8}$, we have $\tan\frac{11\pi}{24} = \tan\left(\frac{\pi}{3} + \frac{\pi}{8}\right) = \dfrac{\sqrt{3} + \sqrt{2} - 1}{1 - \sqrt{3}\left(\sqrt{2} - 1\right)} = \dfrac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1}$, as required.

(ii) Now, in the "spirit" of maths, one might reasonably expect that one should take the given expression, rationalise the denominator (twice) and derive the given answer, along the lines …

$$\frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} = \frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} \times \frac{1 + \sqrt{3} + \sqrt{6}}{1 + \sqrt{3} + \sqrt{6}} = \frac{1 + 2\sqrt{2} + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

However, with a given answer, it is perfectly legitimate merely to multiply across and verify that $\left(\sqrt{3} - \sqrt{6} + 1\right)\left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) = \sqrt{3} + \sqrt{2} - 1$.

(iii) Having got this far, the end is really very clearly signposted. Setting $x = \frac{11\pi}{24}$ in (*) gives

$$\tan\frac{\pi}{48} = \sec\frac{11\pi}{24} - \tan\frac{11\pi}{24} = \sqrt{1 + t^2} - t$$

$$= \sqrt{1 + \left[4 + 2 + 3 + 6 + 4\sqrt{2} + 4\sqrt{3} + 4\sqrt{6} + 2\sqrt{6} + 2\sqrt{12} + 2\sqrt{18}\right]} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$

$$= \sqrt{15 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$