STEP II, 2009, Q2 MS

2(i) It is perfectly possible to differentiate a^(sin[π e^x]) by using the *Chain Rule* (on a function of a function of a function) but simplest to take logs. and use implicit differentiation. Then, setting $\frac{dy}{dx} = 0$ and noting that πe^x and $\ln a$ are non-zero, we are left solving the eqn. $\cos(\pi e^x) = 0$ for the turning points. This gives $e^x = (2n + 1)\frac{1}{2}\pi \Rightarrow \quad x = \ln(n + \frac{1}{2})$, $y = a$ or $\frac{1}{a}$, depending upon whether n is even or odd. Although not actually required at this point, it may be helpful to note at this stage that the evens give maxima while the odds give minima. There is, however, a much more obvious approach to finding the TPs that doesn't require differentiation at all, and that is to use what should be well-known properties of the sine function: namely, that $y = a^{\sin(\pi.\exp x)}$ has maxima when $\sin(\pi e^x) = 1$, i.e. $\pi e^x = (2n + \frac{1}{2})\pi$, and $x = \ln(2n + \frac{1}{2})$ for $n = 0, 1, \ldots$, with $y_{\max} = a$. Similarly, minima occur when $\sin(\pi e^x) = -1$, i.e. $\pi e^x = (2n - \frac{1}{2})\pi$, and $x = \ln(2n - \frac{1}{2})$ for $n = 1, 2, \ldots$, with $y_{\min} = \frac{1}{a}$.

(ii) Using the addition formula for $\sin(A + B)$, and the approximations given, we have

$$\sin(\pi e^x) \approx \sin(\pi + \pi x) = -\sin(\pi x) \approx -\pi x \text{ for small } x,$$

leading to $y \approx a^{-\pi x} = e^{-\pi x \ln a} \approx 1 - \pi x. \ln a$.

(iii) Firstly, we can note that, for $x < 0$, the curve has an asymptote $y = 1$ (as $x \to -\infty$, $y \to 1+$) . Next, for $x > 0$, the curve oscillates between a and $\frac{1}{a}$, with the peaks and troughs getting ever closer together. The work in (i) helps us identify the TPs: the first max. occurs when $n = 0$ at a *negative* value of x [N.B. $\ln(\frac{1}{2}) < 0$] at $y = a$; while the result in (ii) tells us that the curve is approximately negative linear as it crosses the y-axis.

(iv) The final part provides the only really tricky part to the question , and a quick diagram might be immensely useful here. Noting the relevant x-coordinates

$$x_1 = \ln(2k - \tfrac{3}{2}),\ x_2 = \ln(2k - \tfrac{1}{2}),\ \text{and}\ x_3 = \ln(2k + \tfrac{1}{2}),$$

the area is the sum of two trapezia (or rectangle – triangle) , and manipulating

$\ln\left(\frac{4k+1}{4k-3}\right) = \ln\left(\frac{4k-3+4}{4k-3}\right) = \ln\left(1 + \frac{1}{k - \frac{3}{4}}\right)$ leads to the final, given answer.