STEP II, 2009, Q1 MS

1 Both graphs are symmetric in the lines $y = \pm x$, and $x^4 + y^4 = u$ is also symmetric in the x- and y-axes. These facts immediately enable us to write down the coordinates of $B(\beta, \alpha)$, $C(-\alpha, -\beta)$ and $D(-\beta, -\alpha)$. Remember to keep the cyclic order A, B, C, D correct, else this could lead to silly calculational errors later on. The easiest way to show that $ABCD$ is a rectangle is to work out the gradients of the four sides (which turn out to be either 1 or –1) and then note that each pair of adjacent sides is perpendicular using the "product of gradients = –1" result. Working with distances is also a possible solution-approach but, on its own, only establishes that the quadrilateral is a parallelogram. However, the next part requires you to calculate distances anyhow, and we find that CB, DA have length $(\alpha + \beta)\sqrt{2}$ while BA, DC are of length $(\alpha - \beta)\sqrt{2}$. Multiplying these then give the area of $ABCD$ as $2(\alpha^2 - \beta^2)$.

All of this is very straightforward, and the only tricky bit of work comes next. It is important to think of α and β as particular values of x and y satisfying each of the two original equations. It is then clear that $(\alpha^2 - \beta^2)^2 = \alpha^4 + \beta^4 - 2(\alpha^2\beta^2) = u - 2v^2$, so that Area $ABCD = 2\sqrt{u - 2v^2}$. Substituting $u = 81$, $v = 4$ into this formula then gives Area $= 2\sqrt{81 - 2 \times 16} = 14$, which is intended principally as a means of checking that your answer is correct.