STEP II, 2009, Q12 MS

12 (i) This whole question is something of a "one-trick" game, I'm afraid, and relies heavily on being able to spot that X is just half of a normal distribution. The *Standard Normal Distribution* N(0, 1) is given by $P(X \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. Once the connection has been spotted, the accompanying pure maths work is fairly simple, including the sketch of the graph. This is particularly important since the function e^{kx^2} cannot be integrated analytically.

(ii) Substituting $t = 2x$, $dt = 2\,dx$ and equating to $\frac{1}{2}$ (being just the positive half of a normal), gives

$$\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}t^2}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} 2e^{-2x^2}\, dx = \frac{1}{2} \Rightarrow \int_0^{\infty} e^{-2x^2}\, dx = \frac{\sqrt{2\pi}}{4}.$$

Since total probability = 1, we have $\frac{1}{k} = \frac{\sqrt{2\pi}}{4}$ and $k = \frac{4}{\sqrt{2\pi}}$.

(iii) Thereafter, $E(X) = k\int_0^{\infty} xe^{-2x^2}\, dx = k\left[-\frac{1}{4}e^{-2x^2}\right]_0^{\infty} = \frac{1}{4}k = \frac{1}{\sqrt{2\pi}}$.

Also, $E(X^2) = k\int_0^{\infty} x \times x\, e^{-2x^2} dx = k\left\{\left[-\frac{1}{4}xe^{-2x^2}\right]_0^{\infty} + \int_0^{\infty} \frac{1}{4}e^{-2x^2} dx\right\}$ using integration by parts

$$= k\left\{0 + \frac{1}{4} \times \frac{\sqrt{2\pi}}{4}\right\} = \frac{1}{4}.$$

Then $\mathrm{Var}(X) = E(X^2) - E^2(X) = \frac{1}{4} - \frac{1}{2\pi}$ or $\frac{\pi - 2}{4\pi}$.

(iv) For the median, we want to find the value m of x for which $\frac{1}{2} = \frac{4}{\sqrt{2\pi}} \int_0^{m} e^{-2x^2} dx$, and this requires to undo some of the above work in order to be able to use N(0, 1) and the statistics tables provided in the formula book.

$$\frac{1}{2} = \frac{2}{\sqrt{2\pi}} \int_0^{m} 2e^{-2x^2} dx = 2 \times \frac{1}{\sqrt{2\pi}} \int_0^{2m} e^{-\frac{1}{2}t^2} dt = 2\left\{\Phi(2m) - \tfrac{1}{2}\right\} \text{ or } \Phi\left(\tfrac{1}{2}m\right) = \frac{3}{4}$$

Use of the $N(0, 1)$ tables then gives $2m = 0.6745$ (0.675-ish) and $m = 0.337$ or 0.338.