STEP II, 2009, Q11 MS

11 N2L ⇒ $F_T - (n+1)R = (n+1)Ma$, where F_T is the tractive, or driving, force of the engine.

Using $P = F_T\,.\,v$ then gives $a = \dfrac{\frac{P}{v} - (n+1)R}{M(n+1)}$ or $\dfrac{P-(n+1)Rv}{M(n+1)v}$. Note here that, for $a > 0$ we require $P > (n+1)Rv$.

Writing $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$ gives $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{P-(n+1)Rv}{M(n+1)v}$ which is a "variables separable" first-order differential equation: $\dfrac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \mathrm{d}t \Rightarrow \displaystyle\int_0^V \frac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \int_0^T 1\,\mathrm{d}t \quad (=T)$.

Some care is needed to integrate the LHS here, and the simplest approach is to use a substitution such as $s = P-(n+1)Rv$, $\mathrm{d}s = -R(n+1)\,\mathrm{d}v$ to get

$$T = \frac{M}{R}\int \frac{P-s}{s} \times \frac{\mathrm{d}s}{-R(n+1)} = \frac{-M}{(n+1)R^2}\int\left(\frac{P}{s}-1\right)\mathrm{d}s = \frac{-M}{(n+1)R^2}\left[P\ln(s) - s\right]$$

$$= \frac{-M}{(n+1)R^2}\left[P\ln\left(P-(n+1)Rv\right) - \left(P-(n+1)Rv\right)\right]_0^V$$

$$= \frac{-MP}{(n+1)R^2}\left\{\ln\left(P-(n+1)Rv\right) - P + (n+1)Rv - P\ln P + P - 0\right\}$$

$$= \frac{-MP}{(n+1)R^2}\ln\left(\frac{P-(n+1)Rv}{P}\right) - \frac{MV}{R}$$

More careful algebra is still required to manipulate this into a form in which the given approximation can be used:

$$T = \frac{-MP}{(n+1)R^2}\ln\left(1 - \frac{(n+1)Rv}{P}\right) - \frac{MV}{R}$$

$$\approx \frac{-MP}{(n+1)R^2}\left(-\frac{(n+1)Rv}{P} - \frac{1}{2}\left(\frac{(n+1)Rv}{P}\right)^2 \ldots\ldots\right) - \frac{MV}{R}$$

$$= \frac{MV}{R} + \frac{(n+1)MV^2}{2P}\ldots\ldots - \frac{MV}{R}$$

so that $PT \approx \frac{1}{2}(n+1)MV^2$, and this is just the statement of the *Work-Energy Principle*, namely "Work Done = Change in (Kinetic) Energy", in the case when $R = 0$.

When $R \neq 0$, WD against R = WD by engine – Gain in KE ⇒ $(n+1)RX = PT - \frac{1}{2}(n+1)MV^2$.
[Unfortunately, a last-minute change to the wording of the question led to the omission of one of the $(n+1)$s.]