STEP II, 2008, Q9 MS

9 (i) Using a modified version of the trajectory equation (which you are encouraged to have learnt), $y = h + x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$, and substituting in $g = 10$ and $u = 40$ gives

$$y = h + x\tan\alpha - \frac{gx^2}{320}\sec^2\alpha .$$

Setting $x = 20$ and $y = 0$ into this trajectory equation and using one of the well-known *Pythagorean* trig. identities ($\sec^2\alpha = 1 + \tan^2\alpha$) leads to the quadratic equation

$$5t^2 - 80t - (4h - 5) = 0$$

in $t = \tan\alpha$.

[Note that you could have substituted $x = 20$ and $y = -h$ into the unmodified trajectory equation and still got the same result here.]

Solving, using the quadratic formula, and simplifying then gives

$$\tan\alpha = 8 \pm \sqrt{63 + \tfrac{4}{5}h} .$$

We reject $\tan\alpha = 8 + \sqrt{63 + \tfrac{4}{5}h}$, since this gives a very high angle of projection and hence a much greater time for the ball to arrive at the stumps. Now, since α is small, $\cos\alpha \approx 1$, and the time of flight $= \frac{x}{u\cos\alpha} = \frac{1}{2\cos\alpha} \approx \frac{1}{2}$.

(ii) $h > \frac{5}{4}$ for $\tan\alpha = 8 - \sqrt{64 + \varepsilon}\,) < 0$.

(iii) Now $h = 2.5 \Rightarrow \tan\alpha = 8 - \sqrt{64 + 1} = 8 - 8\left(1 + \frac{1}{64}\right)^{1/2}$. The *Binomial Theorem* then allows us to expand the bracket, and it seems reasonable to take just the first term past the 1: $\tan\alpha = 8 - 8\left(1 + \frac{1}{128} + ...\right)$, so that $\tan\alpha \approx -\frac{1}{16}$. [We can ignore the minus sign, since this just tells us that the projection is *below* the horizontal.]

Using $\tan\alpha \approx \alpha$ for small-angles, and converting from radians into degrees using the conversion factor $180/\pi \approx 57.3$ then gives $\alpha \approx 3.6^\circ$.