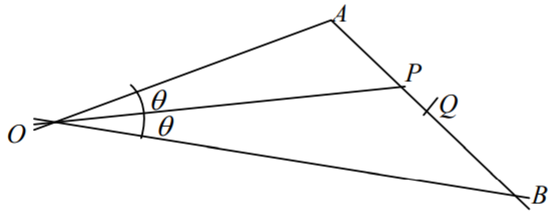
STEP II, 2008, Q8 MS

8 It is never a bad idea to start this sort of question with a reasonably accurate diagram … something along the lines of



The first result is an example of what is known as the *Ratio Theorem*:

$$AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}\ .$$

Alternatively, it can be deduced from the standard approach to the vector equation of a straight line, via $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

Using the scalar product twice then gives

$$\mathbf{a} \bullet \mathbf{p} = \lambda a^2 + (1 - \lambda)(\mathbf{a} \bullet \mathbf{b}) \text{ and } \mathbf{b} \bullet \mathbf{p} = \lambda(\mathbf{a} \bullet \mathbf{b}) + (1 - \lambda)\, b^2\ .$$

Equating these two expressions for $\cos\theta$, $\dfrac{\mathbf{a} \bullet \mathbf{p}}{ap} = \dfrac{\mathbf{b} \bullet \mathbf{p}}{bp}$, re-arranging and collecting up like terms, then gives $ab\{\lambda(a + b) - b\} = \mathbf{a} \bullet \mathbf{b}\ \{\lambda(a + b) - b\}$. There are two possible consequences to this statement, and *both* of them should be considered. Either $ab = \mathbf{a} \bullet \mathbf{b}$, which gives $\cos 2\theta = 1$, $\theta = 0$, $A = B$ and violates the non-collinearity of O, A & B; or the bracketed factor on each side is zero, which gives

$$\lambda = \frac{b}{a + b}\ .$$

However, if you know the *Angle Bisector Theorem*, the working is short-circuited quite dramatically:

$$\frac{AP}{PB} = \frac{OA}{OB} \Rightarrow \frac{(1 - \lambda)(AB)}{\lambda(AB)} = \frac{a}{b} \Rightarrow b - b\lambda = a\lambda \Rightarrow \lambda = \frac{b}{a + b}.$$

Next, $AQ : QB = \lambda : 1 - \lambda \Rightarrow \mathbf{q} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$.

Then

$$OQ^2 = \mathbf{q} \bullet \mathbf{q} = (1 - \lambda)^2\, a^2 + \lambda^2\, b^2 + 2\lambda(1 - \lambda)\ \mathbf{a} \bullet \mathbf{b}$$

and $OP^2 = \mathbf{p} \bullet \mathbf{p} = (1 - \lambda)^2\, b^2 + \lambda^2\, a^2 + 2\lambda(1 - \lambda)\ \mathbf{a} \bullet \mathbf{b}$.

[N.B. This working can also be done by the *Cosine Rule*.]

Subtracting:

$$OQ^2 - OP^2 = (b^2 - a^2)\ [\lambda^2 - (1 - \lambda)^2] = (b^2 - a^2)\ (2\lambda - 1)$$

and, substituting λ in terms of a and b into this expression, gives the required answer

$$= (b - a)(b + a) \times \frac{b - a}{b + a} = (b - a)^2.$$