STEP II, 2008, Q7 MS

7 (i) Differentiating $y = u\sqrt{1+x^2}$ gives $\dfrac{dy}{dx} = u.\dfrac{x}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\dfrac{du}{dx}$; so that

$$\frac{1}{y}\frac{dy}{dx} = xy + \frac{x}{1+x^2} \text{ becomes } \frac{1}{u\sqrt{1+x^2}}\left\{\frac{ux}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{du}{dx}\right\} = xu\sqrt{1+x^2} + \frac{x}{1+x^2}.$$

Simplifying and cancelling the common term on both sides leads to

$$\frac{1}{u}.\frac{du}{dx} = xu\sqrt{1+x^2}.$$

This is a standard form for a first-order differential equation, involving the separation of variables and integration:

$$\int \frac{1}{u^2}.du = \int x\sqrt{1+x^2}\,dx \Rightarrow -\frac{1}{u} = \frac{1}{3}\left(1+x^2\right)^{3/2} (+C).$$

Using $x = 0, y = 1$ $(u = 1)$ to find C leads to the final answer, $y = \dfrac{3\sqrt{1+x^2}}{4-\left(1+x^2\right)^{3/2}}$.

(ii) The key here is to choose the appropriate function of x. If you have really got a feel for what has happened in the previous bit of the question, then this isn't too demanding. If you haven't really grasped fully what's going on then you may well need to try one or two possibilities first. The product that needs to be identified here is $y = u\left(1+x^3\right)^{3/2}$. Once you have found this, the process of (i) pretty much repeats itself.

$$\frac{dy}{dx} = u.x^2\left(1+x^3\right)^{1/2} + \left(1+x^3\right)^{3/2}\frac{du}{dx} \text{ means that } \frac{1}{y}\frac{dy}{dx} = x^2y + \frac{x^2}{1+x^3} \text{ becomes}$$

$$\frac{1}{u}.\frac{du}{dx} = x^2u\left(1+x^3\right)^{3/2}.$$

Separating variables and integrating:

$$\int \frac{1}{u^2}.du = \int x^2\left(1+x^3\right)^{3/2}dx \Rightarrow -\frac{1}{u} = \frac{1}{4}\left(1+x^3\right)^{3/2} (+C);$$

and $x = 0, y = 1$ $(u = 1)$ gives C and the answer $y = \dfrac{4\left(1+x^3\right)^{3/2}}{5-\left(1+x^3\right)^{3/2}}$.

(iii) Note that the question didn't actually require you to simplify the two answers in (i) and (ii), but doing so certainly enables you to have a better idea as to how to generalise the results:

$$y = \frac{(n+1)\left(1+x^n\right)^{3/n}}{(n+2)-\left(1+x^n\right)^{1+3/n}}.$$