STEP II, 2008, Q6 MS

6 (i) Firstly, $\cos x$ has period $2\pi \Rightarrow \cos(2x)$ has period π;

and $\sin x$ has period $2\pi \Rightarrow \sin\left(\frac{3x}{2}\right)$ has period $\frac{4}{3}\pi$.

Then $f(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right)$ has period $4\pi = \text{lcm}(\pi, \frac{4}{3}\pi)$.

(ii) Any approach here is going to require the use of some trig. identity work. The most straightforward is to note that $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ so that $f(x) = 0$ reduces to

$\cos\left(2x + \frac{\pi}{3}\right) = \cos\left(\frac{3x}{2} + \frac{\pi}{4}\right)$, from which it follows that $2x + \frac{\pi}{3} = 2n\pi \pm \left(\frac{3x}{2} + \frac{\pi}{4}\right)$

where n is an integer, using the symmetric and periodic properties of the cosine curve. Taking suitable values of n, so that x is in the required interval, leads to the answers

$x = -\frac{31\pi}{42}$ (from $n = -1$, with the – sign), $x = -\frac{\pi}{6}$ ($n = 0$, with both + and – signs),

$x = \frac{17\pi}{42}$ ($n = 1$, – sign) and $x = \frac{41\pi}{42}$ ($n = 2$, – sign).

Since $x = -\frac{\pi}{6}$ is a repeated root (occurring twice in the above list), the curve of $y = f(x)$ touches the x-axis at this point.

For those who are aware of the results that appear in all the formula books, but which seem to be on the edge of the various syllabuses, that I know by the title of the *Sum-and-Product Formulae*, such as $\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, there is a second straightforward approach available here. For example, noting that

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ gives $\cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{3\pi}{4} - \frac{3x}{2}\right) = 0$ which (from the above identity) then gives $2\cos\left(\frac{x}{4}+\frac{13\pi}{24}\right)\cos\left(\frac{7x}{4}-\frac{5\pi}{24}\right)=0$, and setting each of these two cosine terms equal to zero, in turn, yields the same values of x as before, including the repeat.

(iii) The key observation here is that $y = 2$ if and only if *both* $\cos\left(2x+\frac{\pi}{3}\right)=1$ *and* $\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)=1$, simultaneously. So we must solve

$\cos\left(2x+\frac{\pi}{3}\right)=1 \Rightarrow 2x+\frac{\pi}{3}=0, 2\pi, 4\pi, \ldots$, giving $x=\frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$; and

$\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)=1 \Rightarrow \frac{3x}{2}-\frac{\pi}{4}=\frac{\pi}{2}, \frac{5\pi}{2}, \ldots$, giving $x=\frac{\pi}{2}, \frac{11\pi}{6}, \ldots$.

Both equations are satisfied when $x=\frac{11\pi}{6}$, and this is the required answer.