STEP II, 2008, Q5 MS

5 Using a well-known double-angle formula gives $\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,dx = \int_0^{\pi/2} \frac{2\sin x \cos x}{1+\sin^2 x}\,dx$, and this should suggest an obvious substitution: letting $s = \sin x$ turns this into the integral

$$\int_0^1 \frac{2s}{1+s^2}\,ds\,.$$

This is just a standard log. integral (the numerator being the derivative of the denominator), leading to the answer $\ln 2$.

Alternatively, one could use the identity $\sin^2 x \equiv \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$ to end up with

$$\int_0^{\pi/2} \frac{2\sin 2x}{3-\cos 2x}\,dx.$$

This, again, gives a log. integral, but without the substitution.

A suitable substitution for the second integral is $c = \cos x$, which leads to $\int_0^1 \frac{1}{2-c^2}\,dc$.

Now you can either attack this using partial fractions, or you could look up what is a fairly standard result in your formula booklet. In each case, you get (after a bit of careful log and surd work) $\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$.

Now $(1+\sqrt{2})^5 = 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2} = 41 + 29\sqrt{2}$ (using the binomial theorem, for instance), and

$$41 + 29\sqrt{2} < 99 \;\Leftrightarrow\; 29\sqrt{2} < 58 \;\Leftrightarrow\; \sqrt{2} < 2\,,$$

which is obviously the case. Also, $1.96 < 2 \;\Rightarrow\; 1.4 < \sqrt{2}$. Thereafter, an argument such as

$$2^{1.4} > 1 + \sqrt{2} \;\Leftrightarrow\; 2^7 > (1+\sqrt{2})^5 \;\Leftrightarrow\; 128 > 41 + 29\sqrt{2}$$
$$\Leftrightarrow\; 87 > 29\sqrt{2} \;\Leftrightarrow\; 3 > \sqrt{2}$$

from which it follows that $2^{\sqrt{2}} > 2^{7/5} > 1 + \sqrt{2}$.

Taking logs in this result then gives $\sqrt{2}\,\ln 2 > \ln(1+\sqrt{2}) \;\Rightarrow\; \ln 2 > \frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$; and

$$\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,dx > \int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,dx\,.$$