

STEP II, 2008, Q4 MS

- 4 Differentiating implicitly gives $2\left(x + y \frac{dy}{dx} + ax \frac{dy}{dx} + ay\right) = 0$, from which it follows that

$$\frac{dy}{dx} = -\frac{x + ay}{ax + y} \text{ and hence the gradient of the normal is } \frac{ax + y}{x + ay}.$$

$$\text{Using } \tan(A - B) \text{ on this and } \frac{y}{x} \text{ gives } \tan \theta = \frac{\left| \frac{y}{x} - \frac{ax + y}{x + ay} \right|}{\left| 1 + \frac{y}{x} \times \frac{ax + y}{x + ay} \right|} = \frac{|xy + ay^2 - ax^2 - xy|}{|x^2 + axy + axy + y^2|}.$$

However, we know that $x^2 + y^2 + 2axy = 1$ from the curve's eqn., and so

$$\tan \theta = a|y^2 - x^2|.$$

- (i) Differentiating this w.r.t. x then gives $\sec^2 \theta \frac{d\theta}{dx} = a\left(2y \frac{dy}{dx} - 2x\right)$. Equating this to

$$\text{zero and using } \frac{dy}{dx} = -\frac{x + ay}{ax + y} \text{ from earlier then leads to } a(x^2 + y^2) + 2xy = 0.$$

- (ii) Adding $x^2 + y^2 + 2axy = 1$ and $a(x^2 + y^2) + 2xy = 0$ gives $(1 + a)(x + y)^2 = 1$.

- (iii) However, subtracting these two eqns. instead gives $(1 - a)(y - x)^2 = 1$, and multiplying these two last results together yields $(1 - a^2)(y^2 - x^2)^2 = 1$.

$$\text{Finally, using } \tan \theta = a|y^2 - x^2| \Rightarrow (y^2 - x^2)^2 = \frac{1}{a^2} \tan^2 \theta, \text{ and substituting this}$$

$$\text{into the last result of (iii) then gives the required result: } \tan \theta = \frac{a}{\sqrt{1 - a^2}}. \text{ All that}$$

remains is to justify taking the positive square root, since $\tan \theta$ is |something|, which is necessarily non-negative.



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