STEP II, 2008, Q4 MS

4 Differentiating implicitly gives $2\left(x + y\frac{dy}{dx} + ax\frac{dy}{dx} + ay\right) = 0$, from which it follows that

$\frac{dy}{dx} = -\frac{x+ay}{ax+y}$ and hence the gradient of the normal is $\frac{ax+y}{x+ay}$.

Using $\tan(A - B)$ on this and $\frac{y}{x}$ gives $\tan\theta = \left|\frac{\frac{y}{x} - \frac{ax+y}{x+ay}}{1 + \frac{y}{x} \times \frac{ax+y}{x+ay}}\right| = \left|\frac{xy + ay^2 - ax^2 - xy}{x^2 + axy + axy + y^2}\right|$.

However, we know that $x^2 + y^2 + 2axy = 1$ from the curve's eqn., and so

$$\tan\theta = a\left|y^2 - x^2\right|.$$

(i) Differentiating this w.r.t. x then gives $\sec^2\theta\,\frac{d\theta}{dx} = a\left(2y\frac{dy}{dx} - 2x\right)$. Equating this to zero and using $\frac{dy}{dx} = -\frac{x+ay}{ax+y}$ from earlier then leads to $a(x^2 + y^2) + 2xy = 0$.

(ii) Adding $x^2 + y^2 + 2axy = 1$ and $a(x^2 + y^2) + 2xy = 0$ gives $(1 + a)(x + y)^2 = 1$.

(iii) However, subtracting these two eqns. instead gives $(1 - a)(y - x)^2 = 1$, and multiplying these two last results together yields $(1 - a^2)\,(y^2 - x^2)^2 = 1$.

Finally, using $\tan\theta = a\left|y^2 - x^2\right| \Rightarrow (y^2 - x^2)^2 = \frac{1}{a^2}\tan^2\theta$, and substituting this into the last result of (iii) then gives the required result: $\tan\theta = \frac{a}{\sqrt{1-a^2}}$. All that remains is to justify taking the positive square root, since $\tan\theta$ is | something |, which is necessarily non-negative.