STEP II, 2008, Q3 MS

3 (i) Setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 81x^2 - 54x = 0$ for TPs gives $(0, 4)$ and $\left(\frac{2}{3}, 0\right)$. You really ought to know the shape of such a ("positive") cubic, and it is customary to find the crossing-points on the axes: $x = 0$ gives $y = 4$, and $y = 0$ leads to $x = -1$ and $x = \frac{2}{3}$ (twice). [If you have been paying attention, this latter zero for y should come as no surprise!] The graph now shows that, for all $x \ge 0$, $y \ge 0$; which leads to the required result – $x^2(1 - x) \le \frac{4}{27}$ – with just a little bit of re-arrangement.

In order to prove the result by contradiction (*reduction ad absurdum*), we first assume that all three numbers exceed $\frac{4}{27}$. Then their product

$$bc(1-a)ca(1-b)ab(1-c) > \left(\tfrac{4}{27}\right)^3.$$

However, this product can be re-written in the form

$$a^2(1-a).\ b^2(1-b.)\ c^2(1-c),$$

and the previous result guarantees that $x^2(1-x) \le \frac{4}{27}$ for each of a, b, c, from which it follows that

$$a^2(1-a).\ b^2(1-b.)\ c^2(1-c) \le \left(\tfrac{4}{27}\right)^3,$$

which is the required contradiction. Hence, at least one of the three numbers $bc(1-a)$, $ca(1-b)$, $ab(1-c)$ is less than, or equal to, $\frac{4}{27}$.

(ii) Drawing the graph of $y = x - x^2$ (there are, of course, other suitable choices, such as $y = (2x-1)^2$ for example) and showing that it has a maximum at $\left(\frac{1}{2}, \frac{1}{4}\right)$ gives

$$x(1-x) \le \tfrac{1}{4} \text{ for all } x.$$

The assumption that $p(1-q)$, $q(1-p) > \frac{1}{4} \Rightarrow p(1-p).q(1-q) > \left(\frac{1}{4}\right)^2$.

However, we know that $x(1-x) \le \frac{1}{4}$ for each of p and q, and this gives us that

$$p(1-p).q(1-q) \le \left(\tfrac{1}{4}\right)^2.$$

Hence, by contradiction, at least one of $p(1-q)$, $q(1-p) \le \frac{1}{4}$.