STEP II, 2008, Q3 EC

Q3 This, the third most popular question on the paper, producing a mixed bag of responses. It strikes me that, although the A-level specifications require candidates to understand the process of *proof by contradiction*, this is never actually tested anywhere by any of the exam. boards. Nonetheless, it was very pleasing to see that so many candidates were able to grasp the basic idea of what to do, and many did so very successfully. The impartial observer might well note that the situation in (i) is very much tougher (in terms of degree) than that in (ii). However, candidates were very much more closely guided in (i) and then left to make their own way in (ii).

Apart from the standard, expected response to (i) – see the *SOLUTIONS* document for this – many other candidates produced a very pleasing alternative which they often dressed up as *proof by contradiction* but which was, in fact, a direct proof. It was, however, so mathematically sound and appealing an argument (and a legitimate imitation of a *p by c*) that we gave it all but one of the marks available in this part of the question. It ran like this:

Suppose w.l.o.g. that $0 < a \le b \le c < 1$.

Then $ab(1 - c) \le b^2(1 - b) \le \frac{4}{27}$ by the previous result

(namely $x^2(1 - x) \le \frac{4}{27}$ for all $x \ge 0$).

QED.

[Note that we could have used $ab(1 - c) \le c^2(1 - c) \le \frac{4}{27}$ also.]

It has to be said that most other inequality arguments were rather poorly constructed and unconvincing, leaving the markers with little option but to put a line through (often) several pages of circular arguments, faulty assumptions, dubious conclusions, and occasionally correct statements with either no supporting reasoning or going nowhere useful.

There was one remarkable alternative which was produced by just a couple of candidates (that I know of) and is not included in the *SOLUTIONS* because it is such a rarity. However, for those who know of the *AM – GM Inequality*, it is sufficiently appealing to include it here for novelty value. It ran like this:

Assume that $bc(1 - a),\ ca(1 - b),\ ab(1 - c) > \frac{4}{27}$.

Using the previous result, we have $a^2(1 - a),\ b^2(1 - b),\ c^2(1 - c) \le \frac{4}{27}$.

Then, since all terms are positive, it follows that $a^2 \le bc,\ b^2 \le ca,\ c^2 \le ab$ so that

$$a^2 + b^2 + c^2 \le bc + ca + ab. \text{ (*)}$$

However, by the *AM – GM Inequality* (or directly by the *Cauchy-Schwarz Inequality*),

$$a^2 + b^2 \ge 2ab,\ b^2 + c^2 \ge 2bc \text{ and } c^2 + a^2 \ge 2ca.$$

Adding and dividing by two then gives $a^2 + b^2 + c^2 \ge ab + bc + ca$, which contradicts the conclusion **(*)**, etc., etc.