STEP II, 2008, Q2 MS

2 The correct partial fraction form for the given algebraic fraction is

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{Cx+D}{1+x^2},$$

although these *can* also be put together in other correct ways that don't materially hinder the progress of the solution. The standard procedure now is to multiply throughout by the denominator of the LHS and compare coefficients or substitute in suitable values: which leads to $A = \frac{1}{2}$, $B = 1$, $C = \frac{1}{2}$ and $D = -\frac{1}{2}$.

In order to apply the binomial theorem to these separate fractions, we now use index notation to turn

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv A(1-x)^{-1} + B(1-x)^{-2} + Cx(1+x^2)^{-1} + D(1+x^2)^{-1}$$

into the infinite series

$$\tfrac{1}{2}\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(n+1)x^n + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n+1} - \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

It should be clear at this point that the last two of these series have odd/even powers only, with alternating signs playing an extra part. The consequence of all this is that we need to examine cases for n modulo 4; i.e. depending upon whether n leaves a remainder of 0, 1, 2 or 3 when divided by 4.

For $n \equiv 0 \pmod 4$, the coefft. of x^n is $\frac{1}{2} + n + 1 + 0 - \frac{1}{2} = n + 1$;

A1 for $n \equiv 1 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + \frac{1}{2} - 0 = n + 2$;

A1 for $n \equiv 2 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + 0 + \frac{1}{2} = n + 2$;

A1 for $n \equiv 3 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 - \frac{1}{2} + 0 = n + 1$.

For the very final part of the question, we note that $\frac{11000}{8181} = \frac{1.1}{0.9^2 \times 1.01}$, is a cancelled form of our original expression, with $x = 0.1$. (N.B. $|x| < 1$ assures the convergence of the infinite series forms). Substituting this value of x into

$$1 + 3x + 4x^2 + 4x^3 + 5x^4 + 7x^5 + 8x^6 + 8x^7 + 9x^8 + \ldots$$

then gives 1.344 578 90 to 8dp.