STEP II, 2008, Q1 MS

1 (i) Given $(x_{n+1}, y_{n+1}) = (x_n^{\ 2} - y_n^{\ 2} + 1,\ 2x_n y_n + 1)$, it is easier to remove the subscripts and set $x^2 - y^2 + 1 = x$ and $2xy + 1 = y$. Then, identifying the y's (or x's) in each case, gives $y^2 = x^2 - x + 1$ and $y = \dfrac{1}{1-2x}$. Eliminating the y's leads to a polynomial equation in x; namely, $4x^4 - 8x^3 + 9x^2 - 5x = 0$.

Noting the obvious factor of x, and then finding a second linear factor (e.g. by the factor theorem) leads to $x(x-1)(4x^2 - 4x + 5) = 0$. Here, the quadratic factor has no real roots, since the discriminant, $\Delta = 4^2 - 4.4.5 = -64 < 0$. [Alternatively, one could note that $4x^2 - 4x + 5 \equiv (2x-1)^2 + 4 > 0\ \forall x$.]

The two values of x, and the corresponding values of y, gained by substituting these x's into $y = \dfrac{1}{1-2x}$, are then $(x, y) = (0, 1)$ and $(1, -1)$

(ii) Now $(x_1, y_1) = (-1, 1) \Rightarrow (x_2, y_2) = (a, b)$ and $(x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)$. Setting both $a^2 - b^2 + a = -1$ and $2ab + b + 2 = 1$, so that the third term is equal to the first, and identifying the b's in each case, gives $b^2 = a^2 + a + 1$ and $b = \dfrac{-1}{1+2a}$.

One could go about this the long way, as before. However, it can be noted that the algebra is the same as in (i), but with $a = -x$ and $b = -y$. Either way, we obtain the two possible solution-pairs: $(a, b) = (0, -1)$ and $(-1, 1)$.

However, upon checking, the solution $(-1, 1)$ actually gives rise to a constant sequence (and remember that the working only required the third term to be the same as the first, which doesn't preclude the possibility that it is also the same as the second term!), so we find that there is in fact just the one solution: $(a, b) = (0, -1)$.