STEP II, 2008, Q13 MS

13 (i) $p(\text{B in bag P}) = p(\text{B not chosen draw 1}) + p(\text{B chosen draw 1 and draw 2})$

$$= \left(1 - \frac{k}{n}\right) + \frac{k}{n} \times \frac{k}{n+k}$$

$$= \frac{1}{n(n+k)}\left((n-k)(n+k) + k^2\right)$$

$$= \frac{n}{n+k}$$

This has its maximum value of 1 for $k = 0$, and for no other values of k. Since $p = 1 - \dfrac{k}{n+k} \le 1$ and for $k = 0, p = 1$ but $k > 0$ for all $p < 1$).

(ii) $p(\text{Bs in same bag}) = p(\text{B}_1 \text{ chosen on D}_1 \text{ and neither chosen on D}_2)$
$+ p(\text{B}_1 \text{ chosen on D}_1 \text{ and both chosen on D}_2)$
$+ p(\text{B}_1 \text{ not chosen on D}_1 \text{ and B}_2 \text{ chosen on D}_2)$

$$= \frac{k}{n} \times \frac{{}^{n+k-2}C_k}{{}^{n+k}C_k} + \frac{k}{n} \times \frac{{}^{n+k-2}C_{k-2}}{{}^{n+k}C_k} + \left(1 - \frac{k}{n}\right) \times \frac{k}{n+k}$$

Notice that, although the ${}^nC_r$ terms *look* very clumsy, they are actually quite simple once all the cancelling of common factors has been undertaken.

$$= \frac{k}{n} \times \frac{n(n-1)}{(n+k)(n+k-1)} + \frac{k}{n} \times \frac{k(k-1)}{(n+k)(n+k-1)} + \frac{k(n-k)}{n(n+k)}$$

$$= \frac{k}{n}\left\{\frac{n^2 - n + k^2 - k + (n^2 + nk - n - nk - k^2 + k)}{(n+k)(n+k-1)}\right\}$$

$$= \frac{2k(n-1)}{(n+k)(n+k-1)}$$

Differentiating this expression gives

$$\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{(n^2 + 2nk + k^2 - n - k) \times 2(n-1) - 2k(n-1) \times (2n + 2k - 1)}{[(n+k)(n+k-1)]^2}$$

$= 0$ when $n^2 + 2nk + k^2 - n - k = 2nk + 2k^2 - k$ since $n > 2,\ n - 1 \ne 0$

$\Rightarrow k^2 = n(n-1)$.

Now there is nothing that guarantees that k is going to be an integer (quite the contrary, in fact), so we should look to the integers either side of the (positive) square root of $n(n-1)$:

$$k = \left[\sqrt{n(n-1)}\right] \quad \text{and} \quad k = \left[\sqrt{n(n-1)}\right] + 1 .$$

In fact, since $n^2 - n = (n - ½)^2 - ¼$, $\left[\sqrt{n^2 - n)}\right] = n - 1$ and we find that,

when $k = n - 1$, $p = \dfrac{2(n-1)^2}{(2n-1)2(n-1)} = \dfrac{n-1}{2n-1}$

and when $k = n$, $p = \dfrac{2n(n-1)}{(2n)(2n-1)} = \dfrac{n-1}{2n-1}$ also,

and $k = n - 1, n$ are the two values required.