STEP II, 2008, Q12 MS

12 Clearly, $X \in \{0, 1, 2, 3\}$ and working out the corresponding probabilities is a good thing to do at some point in this question (although it can, of course, be done numerically later when a value for p has been found).

$p(X=0) = (1-p)(1-\frac{1}{3}p)(1-p^2)$

$p(X=1) = p(1-\frac{1}{3}p)(1-p^2) + (1-p)\frac{1}{3}p(1-p^2) + (1-p)(1-\frac{1}{3}p)p^2$

$= p(1-p)(\frac{4}{3} + \frac{5}{3}p - p^2)$

$p(X=2) = p.\frac{1}{3}p(1-p^2) + p(1-\frac{1}{3}p)p^2 + (1-p)\frac{1}{3}p.p^2$

$= \frac{1}{3}p^2(1+4p-3p^2)$

$p(X=3) = \frac{1}{3}p^4$

[Of course, one of these could be deduced on a (1 – the sum of the rest) basis, but that can always be left as useful check on the correctness of your working, if you so wish.]

Then $E(X) = \sum x.p(x) = 0 + p(1-p)(\frac{4}{3} + \frac{5}{3}p - p^2) + \frac{2}{3}p^2(1+4p-3p^2) + p^4$

$= \frac{4}{3}p + p^2$

Alternatively, if you have done a little bit of expectation algebra, it is clear that

$$E(X) = \sum E(X_i) = p + \frac{1}{3}p + p^2 = \frac{4}{3}p + p^2.$$

Equating this to $\frac{4}{3} \Rightarrow 0 = 3p^2 + 4p - 4 \Rightarrow 0 = (3p-2)(p+2)$, and since $0 < p < 1$ it follows that $p = \frac{2}{3}$.

In the final part, you will need either (p_0 and p_1) or (p_2 and p_3):

$$p_0 = \frac{35}{243} \text{ and } p_1 = \frac{108}{243} \text{ or } \quad p_2 = \frac{84}{243} \text{ and } p_3 = \frac{16}{243}.$$

Next, a careful statement of cases is important (with, I hope, obvious notation):

p(correct pronouncement) = p(G and ≥ 2 judges say G) + p(NG and ≤ 1 judges say G)

$$= t \cdot \frac{100}{243} + (1-t) \cdot \frac{143}{243} = \frac{143 - 43t}{243}$$

Equating this to $\frac{1}{2}$ and solving for $t \Rightarrow 243 = 286 - 86t \Rightarrow 86t = 43 \Rightarrow t = \frac{1}{2}$.

Alternatively, let p(King pronounces guilty) = q.

Then "King correct" = "King pronounces guilty and defendant *is* guilty"

or "King pronounces not guilty and defendant *is* not guilty"

so that p(King correct) = $qt + (1-q)(1-t)$.

Setting $qt + (1-q)(1-t) = \frac{1}{2} \Leftrightarrow (2q-1)(2t-1) = 0$, and since q is not identically equal to $\frac{1}{2}$, $t = \frac{1}{2}$.