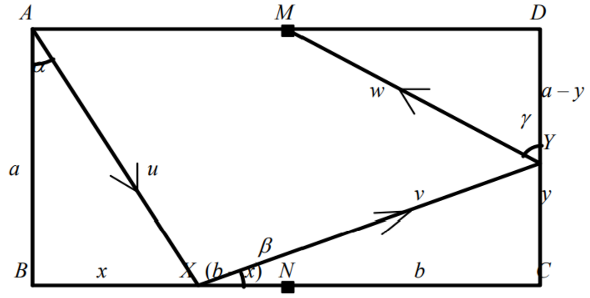
STEP II, 2008, Q10 MS

10 On this sort of question, a good, clear diagram is almost essential, even when it is not asked-for.



(i) The two fundamental principles involved in collisions are the *Conservation of Linear Momentum* (*CLM*) and *Newton's Experimental Law of Restitution* (*NEL* or *NLR*).
For the collision at *X*, applying $CLM \parallel BC \quad \Rightarrow mu \sin\alpha = mv\cos\beta$
.. $NEL \quad \Rightarrow eu\cos\alpha = v\sin\beta$

Dividing these two gives $\tan\beta = e\cot\alpha$ or $\tan\alpha\tan\beta = e$.
At *Y*, "similarly", we have $\tan\beta\tan\gamma = e$. Hence $\alpha = \gamma$ (since all angles are acute).

(ii) A good approach here is to use similar Δs, and a bit of sensible labelling is in order (see the diagram). Let $BX = x$ $(XN = b - x)$ and $CY = y$ $(DY = a - y)$. Then

$$\tan\alpha = \frac{x}{a}, \quad \tan\beta = \frac{y}{2b - x}, \quad \tan\gamma = \frac{b}{a - y}.$$

Using $\alpha = \gamma$ to find (e.g) y in terms of $a, b, x \Rightarrow ax - xy = ab \Rightarrow y = \dfrac{a(x-b)}{x}$.

Next, we use the result $\tan\alpha\tan\beta = e$ from earlier to get x in terms of a and b:

$$\frac{x}{a} \times \frac{a(x-b)/x}{2b - x} = e \Rightarrow x - b = 2be - ex \Rightarrow x = \frac{b(1+2e)}{1+e},$$

from which it follows that $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)}$.

At this stage, some sort of inequality argument needs to be considered, and a couple of obvious approaches might occur to you.

I $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{b}{a} + \dfrac{be}{a(1+e)} > \dfrac{b}{a}$ and $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{2b}{a} - \dfrac{be}{a(1+e)} < \dfrac{2b}{a}$

give $\dfrac{b}{a} < \tan\alpha < \dfrac{2b}{a}$; and the shot is possible, with the ball striking *BC* between *N* and *C*, whatever the value of e .

II As $e \to 0$, $\tan\alpha \to \dfrac{b}{a}+$ and as $e \to 1$, $\tan\alpha \to \dfrac{3b}{2a}-$, so that

$\dfrac{b}{a} < \tan\alpha < \dfrac{3b}{2a}$; and the shot is possible, with the ball striking *BC* between *N* and the midpoint of *NC*, whatever the value of *e*.

(iii) There are two possible approaches to this final part. The first, much longer version, involves squaring and adding the eqns. for the collision at X, and then again at Y, to get

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad \text{and} \quad w^2 = v^2(\sin^2\beta + e^2\cos^2\beta).$$

Now, noting that the initial KE = $\frac{1}{2}mu^2$ and the final KE = $\frac{1}{2}mw^2$, the fraction of

KE lost is $\dfrac{\frac{1}{2}mu^2 - \frac{1}{2}mw^2}{\frac{1}{2}mu^2} = 1 - \dfrac{w^2}{u^2} = 1 - (\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$

$$= 1 - \frac{\tan^2\alpha + e^2}{\sec^2\alpha} \times \frac{\tan^2\beta + e^2}{\sec^2\beta}.$$

From here, we use $\tan\alpha\tan\beta = e$ and $\sec^2\alpha = 1 + \tan^2\alpha$ to get

$$1 - \frac{t^2 + e^2}{1 + t^2} \times \frac{e^2/t^2 + e^2}{1 + e^2/t^2} = 1 - \frac{t^2 + e^2}{1 + t^2} \times \frac{e^2(1 + t^2)/t^2}{(t^2 + e^2)/t^2} = 1 - e^2, \text{ as required.}$$

However, it is very much quicker to note the following:

At X, the ↑-component of the ball's velocity becomes $e \times$ initial ↑-component ,

and

at Y, the →-component of the ball's velocity becomes $e \times$ initial →-component.

Hence its final velocity is eu and the fraction of the KE lost is then

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = 1 - e^2.$$