

STEP II, 2007, Q7

- 7 A function $f(x)$ is said to be concave on some interval if $f''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $f(x)$ be concave on a given interval and let x_1, x_2, \dots, x_n lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n} \sum_{k=1}^n f(x_k) \leq f\left(\frac{1}{n} \sum_{k=1}^n x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \dots = x_n$. You may use this result without proving it.

- (i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

- (ii) By choosing a suitable function f , prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \leq \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers t_1, t_2, \dots, t_n .

Hence:

- (a) show that $x^4 + y^4 + z^4 + 16 \geq 8xyz$, where x, y and z are any positive numbers;
 (b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.



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