STEP II, 2007, Q7

7 A function $\mathrm{f}(x)$ is said to be concave on some interval if $\mathrm{f}''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $\mathrm{f}(x)$ be concave on a given interval and let x_1, x_2, ..., x_n lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{f}(x_k) \leqslant \mathrm{f}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$. You may use this result without proving it.

(i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}.$$

(ii) By choosing a suitable function f, prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \leqslant \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers t_1, t_2, ..., t_n.

Hence:

(a) show that $x^4 + y^4 + z^4 + 16 \geqslant 8xyz$, where x, y and z are any positive numbers;

(b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.