STEP II, 2007, Q5 MS

Q5 Part (i) is a standard opener using compositions of functions, and the algebra shouldn't prove too demanding if you're careful. Again, simplified answers at each stage are most helpful for successful further progress through a question like this. The sequence of powers of f turns out to be periodic with period 3, and so f^{2007} isn't quite the big ask that it might seem to be at first sight.

As you're told what to do in (ii), it is just a case of being careful in establishing the relationship. A grasp of the process of mathematical induction is an essential requirement for STEP II, even if it is no longer on single Maths syllabuses elsewhere, and this could be used in this case. An informal inductive proof was perfectly acceptable also, although it was equally acceptable to establish the cases for $n = 1$, 2 and 3 and then point out that the periodicity of the tan function guarantees the rest.

Now, part (iii) offers something a little more demanding. The simple approach involves spotting that the use of $t = \sin\theta$ gives $\sqrt{1-t^2} = \cos\theta$, and then a similar inductive argument to (ii)'s will lead to an admittedly unappealing but otherwise simple result for g^n in a $\sin(A+B)$ kind of way. However, if instead you note that $\sqrt{1-t^2}$ denotes the **positive** square-root of $1-t^2$, which may actually be $-\cos\theta$ for some values of θ (and hence t). Thus, in fact, g^2 *can* turn out to be just x again, so that the sequence $\{g, g^2, g^3, \ldots\}$ turns out to be oscillating (i.e. periodic with period 2). If you proceed further down this route, exploring which parts of g's domain give what "powers" of g, you get very interesting results which may be worth discussion, but were not expected under examination conditions here.

Answers: (i) $f^2(x) = \dfrac{x-\sqrt{3}}{1+\sqrt{3}\,x}$, $f^3(x) = x$; $f^{2007}(x) = x$.

(iii) Answer 1: $g^n(t) = \sin(\sin^{-1} t + \frac{n\pi}{6})$; Answer 2: $g^n(t) = \begin{cases} g(t) & n \text{ odd} \\ t & n \text{ even} \end{cases}$.