STEP II, 2007, Q2 MS

Q2 It is fairly obvious that $x = p$ and $x = q$ are the two roots of the equation $\frac{dy}{dx} = 0$, which means that the derivative is a multiple of $(x - p)(x - q)$. Comparing the two then immediately gives b and c in terms of p and q. The sketch is a standard (positive) cubic, through the origin, with its two TPs in the first quadrant. Unintentionally, there are two possible candidates for the region $\boldsymbol{R}$, since the setters omitted to consider the one of them. Almost all candidates taking this paper identified the intended region, and this was because the question tries to get you to focus on the area around the point of inflection, which you are asked to mark on the diagram.

In (iii), m and n are simply the y-coordinates of the points corresponding to $x = p$ and q (respectively), and by this point you should know the curve's equation (in terms of p and q rather than b and c). Notice that $y(m)$ involves the extra qs and $y(n)$ involves extra ps, so the difference may just involve lots of $(q - p)$s, and the answer effectively tells you this much also. It may help in the working, both now and later, if you exploit this difference as much as possible.

Before embarking on the final part of the question, it would benefit you greatly to take a momentary pause and think about how the various bits of the question hang together. You were earlier asked to describe the symmetry of the cubic, and this was not just an idle bit of space-filler on the setter's part. Rather, it was an attempt to force you into recognising that the area of the region $\boldsymbol{R}$ can be found by means other than integration. Ignoring the coordinate axes on the diagram, and looking at the lines $x = p$, $x = q$, $y = n$ and $y = m$, you will see a nice rectangle appearing in the middle of the page. Because of the symmetry of the cubic, $\boldsymbol{R}$ is something to do with this rectangle, and this fact pretty much allows you to write the answer straight down, using the answer to (iii). On the other hand, if you want to do it by integration (as most candidates did) …..

And if you feel up to an algebraic challenge, see if you can work out, by integration, the area of the other possible region $\boldsymbol{R}$ – which also turns out – rather surprisingly, I felt – to be a rational multiple of $(q - p)^4$.

Answers: (i) $b = 3(p + q)$, $c = 6pq$; (ii) (two-fold) rotational symmetry about the P of I.