STEP II, 2006, Q5 MS

Q5 The crucial observation here is that the integer-part (or *INT* or "floor") function is a whole number. Thus, when drawing the graphs, the two curves must coincide at the left-hand (integer) endpoints of each unit range, with the second curve slowly falling behind in the first instance, and remaining at the integer level in the second. Note that the curves with the *INT* function-bits in them will jump at integer values, and you should not therefore join them up at the right-hand ends (to form a continuous curve).

The easiest approach in (i) is not to consider $\int y_1 \, dx - \int y_2 \, dx$ (i.e. separately), but rather $\int (y_1 - y_2) dx$. This gives a multiple of $x - [x]$ to consider at each step, and this simply gives a series of "unit" right-angled triangles of area $\frac{1}{2}$ to be summed.

In (ii), several possible approaches can be used, depending upon how you approached (i). If you again focus on the difference in area across a representative integer range, then you end up having to sum $k + \frac{11}{6}$ from $k = 1$ to $k = n - 1$. Otherwise, there is some integration (for the continuous curves) and some summation (for the integer-part lines) to be done, which may require the use of standard summation results for $\sum k$ and $\sum k^2$.

Answers: (i) $\frac{3}{2} n(n-1)$.