STEP II, 2006, Q3 MS

Q3 If you fail to notice that $\frac{1}{5+\sqrt{24}} = 5 - \sqrt{24}$, then this question is going to be a bit of a non-starter for you. The idea of conjugates, from the use of the *difference of two squares*, should be a familiar one. As is the *binomial theorem*, which you can now use to expand both $(5+\sqrt{24})^4$ and $(5-\sqrt{24})^4$. When you do this, you will see that all the $\sqrt{24}$ bits cancel out, to leave you with an integer. For the next part, some fairly simple inequality observations, such as

$$20.25 < 24 < 25 \Rightarrow 4.5 < \sqrt{24} < 5 \text{ and } 2 \times 100 = 200 < 208 = 11 \times 19 \Rightarrow \frac{2}{19} < \frac{11}{100}$$

help to establish the required results. It follows that $0.1^4 < (5-\sqrt{24})^4 < 0.11^4$ and the difference between the integer and $(5+\sqrt{24})^4$ is this small number, which lies between …..

For part (ii), it is simply necessary to mimic the work of part (i) but in a general setting, again starting with the key observations that $\frac{1}{N+\sqrt{N^2-1}} = N - \sqrt{N^2-1}$ and that the binomial expansions for $\left(N+\sqrt{N^2-1}\right)^k + \left(N-\sqrt{N^2-1}\right)^k$ will lead to the cancelling of all surd terms, to give an integer, M say. Now $\left(N-\sqrt{N^2-1}\right)^k$ is positive, and the reciprocal of a number > 1, so $\left(N-\sqrt{N^2-1}\right)^k \to 0+$ as $k \to \infty$. Also,

$$2N - \tfrac{1}{2} < N + \sqrt{N^2-1} < 2N \Rightarrow \frac{1}{2N-\frac{1}{2}} > N - \sqrt{N^2-1} > \frac{1}{2N}.$$

Thus $\left(N+\sqrt{N^2-1}\right)^k = M - \left(N-\sqrt{N^2-1}\right)^k$ differs from an integer (M) by less than $\left(\frac{1}{2N-\frac{1}{2}}\right)^k = (2N-\tfrac{1}{2})^{-k}$.

Answers: (i) 9601.9999