STEP II, 2006, Q2 MS

Q2 The formula books give a series for e^x. Setting $x = 1$ then gives you e as the limit of an infinite sum of positive terms, and the sum of the first four of these will then provide a lower bound to its value.

In the next part, you (again) can provide a perfectly sound argument for the required result without having to resort to a formally inductive one (although one would be perfectly valid, of course). Noting firstly that $4! = 24 > 16 = 2^4$, $(n + 4)!$ consists of the product of $4!$ and n positive integers, each greater than 2; while 2^{n+4} consists of 16 and a further n factors of 2. Since each term in the first number is greater than the corresponding term in the second, the result follows. [Alternatively, $4! > 2^4$ and $n! > 2^n \Rightarrow (n + 1)! = (n + 1) \times n! > 2 \times n!$ (since $n > 4$) $> 2 \times 2^n$ (by hypothesis) $= 2^{n+1}$, and proof follows by induction.] Now, adding the terms in the expansion for e **beyond** the cubed one, and noting that each is less than a corresponding power of $\frac{1}{2}$ using the result just established, gives $e < \frac{8}{3}$ + the sum-to-infinity of a convergent GP.

There are two common methods for showing that a stationary value of a curve is a max. or a min. One involves the second derivative evaluated at the point in question.

There are several drawbacks involved with this approach. One is that you have to differentiate twice (which is ok with simple functions). A second is that you need to know the exact value(s) of the variable being substituted (which isn't the case here). Another is that the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ doesn't necessarily tell you what is happening to the curve. (Think of the graph of $y = x^4$, which has $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at the origin, yet the stationary point here ***is*** a minimum!)

Thus, it is the other approach that you are clearly intended to use on this occasion. This examines the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ slightly to each side of the point in question. When $x = \frac{1}{2}$, using $e < \frac{67}{24}$ shows …..; at $x = 1$, using $e > \frac{8}{3}$ shows …..; and at $x = \frac{5}{4}$, we can use any suitable bound for e, such as $e < 3$ for instance, to show that …..

Finally, since the answers are given in the question, it is important to state carefully the reasoning that supports these answers.