STEP II, 2006, Q1 MS

Q1 If you read through at least part (i) of the question, you will see that it is necessary to work with u_1, u_2, u_3 and u_5 (and hence, presumably – as the sequence is defined recursively – with u_4 also). Although it is not the only way to go about the problem, it makes sense to work each of these terms out first. Each will be an expression involving k and should ideally be simplified as you go. Thus, $u_1 = 2$ gives $u_2 = k - 18$, $u_3 = k - \frac{36}{k-18} = \frac{k^2 - 18k - 36}{k-18}$, etc. Then, for (a), $u_2 = 2$; for (b), $u_3 = 2$; and, for (c), $u_5 = 2$. Each result leads to a polynomial equation (of increasing orders) to be solved. Finally, you need to remember that, in the case of (c) for instance, of the four solutions given by the resulting equation, two of them must have arisen already in parts (a) and (b) – you'll see why if you think about it for a moment. Ideally, you would see this beforehand, and then this fact will help you factorise the quartic polynomial by the *factor theorem*.

A simple line of reasoning can be employed to establish the first result in (ii) without the need for a formal inductive proof. If $u_n \geq 2$, then $u_{n+1} = 37 - \frac{36}{u_n} \geq 37 - \frac{36}{2} = 19 > 2$. Since $u_1 = 2$, it follows that all terms of the sequence are ≥ 2. In fact, most of them are much bigger than this. Then, for the final part of the question, the informal observation that, eventually, all terms effectively become equal is all that is required. Setting $u_{n+1} = u_n = l$ (say) leads to a quadratic, with two roots, one of which is obviously less than 2 and can therefore be rejected.

Answers: (i) $k =$ (a) 20; (b) 0; (c) $\pm 6\sqrt{2}$. (ii) 36.