# STEP II, 2005, Q8 MS

*Q8* Separation of variables will lead to

$$A - 1/y = \int x^3(1+x^2)^{-5/2}\,dx,$$

where $A$ is a constant. There are several possible strategies for the evaluation of the integral on the right; by parts, or by any of such substitutions as $w = 1 + x^2$, $x = \tan t$, $x = \sinh v$. One way or another, the result of this integration correctly carried out will lead to the equivalent of

$$A - 1/y = -(1/3)x^2(1+x^2)^{-3/2} - (2/3)(1+x^2)^{-1/2}.$$

Use of the initial condition $y(0) = 1$ will then show that $A = 1/3$ and the required result follows at once.

$1/y = 1/3 + (2+3x^2)/[3(1+x^2)^{3/2}]$.

To obtain the required approximation for $y$ for large postive $x$, first write

$$1/y \approx 1/3 + (2+3x^2)(1 - 3/2x^2)/3x^3,$$

from which it follows that $1/y = 1/3 + 1/x + O(1/x^3)$ and this can easily be worked to the displayed approximation for $y$.

For the sketch, the main features are that it has a zero gradient at $x = 0$ and that for $x > 0$, it is monotonically increasing and has exactly one point of inflexion and that it is asymptotic to the line $y = 3$.

The two given differential equations are related by $y = z^2$. Thus it is unnecessary to solve the second differential equation independently of the first. In any case, the question does not require a formal solution for $z$. Nevertheless, it is helpful to obtain the approximation $z \approx \pm\sqrt{3} \mp 3\sqrt{3}/2x$ from $y \approx 3 - 9/x$, for large positive $x$.

All the main features of the $x - z$ sketch may be derived from those listed above for the $x - y$ sketch. The two curves which make up the sketch of $z$ are reflections of each other in the $x-$ axis. They start at $(0, \pm 1)$ and are asymptotic to the lines $z = \pm\sqrt{3}$.

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