STEP II, 2005, Q8

8 For $x \geqslant 0$ the curve C is defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3y^2}{(1+x^2)^{5/2}}$$

with $y = 1$ when $x = 0$. Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}\,.$$

Draw a sketch of C.

On a separate diagram draw a sketch of the two curves defined for $x \geqslant 0$ by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x^3z^3}{2(1+x^2)^{5/2}}$$

with $z = 1$ at $x = 0$ on one curve, and $z = -1$ at $x = 0$ on the other.