STEP II, 2005, Q7 MS

Q7 (i) The absence of a **k** component in the specification of the locus of P, shows immediately that its motion takes place in the plane $z = 0$, i.e. in the $x - y$ plane. Also, it is obvious that $x^2 + y^2 = 1$. Hence P describes a circle centre O and radius 1 in the $x - y$ plane.

For the locus of Q, it is helpful to write $x = (3/2)\cos(t+\pi/4)$, $y = 3\sin(t+\pi/4)$, $z = (3\sqrt{3}/2)\cos(t+\pi/4)$. It is then evident that $\sqrt{3}x - z = 0$ and so this defines the plane in which the motion of Q takes place. Furthermore, it is clear that $x^2 + y^2 + z^2 = 9$ which shows that the distance of Q from O is constant and equal to 3. Hence Q describes the circle centre O and radius 3 in the plane $\sqrt{3}x - z = 0$.

(ii) Use of the scalar product leads to $\cos\theta = |(1/2)\cos t\cos(t+\pi/4) + \sin t\sin(t+\pi/4)| = \ldots = |3/4\sqrt{2} - (1/4)\cos(2t+\pi/4)|$.

(iii) From the result just obtained, it is immediate that $\theta \geq \pi/4 \Rightarrow -1/\sqrt{2} \leq 3/4\sqrt{2} - c/4 \leq 1/\sqrt{2}$ $\quad (c \equiv \cos(2t+\pi/4) \Rightarrow -1/\sqrt{2} \leq c \leq 7/\sqrt{2}$,

and as $c \leq 1$, then c is restricted to the interval $-1/\sqrt{2} \leq c \leq 1$ so that $t \notin [\pi/4, \pi/2]$ and $t \notin [5\pi/4, 3\pi/2]$ are required. Hence $T = 3\pi/2$