STEP II, 2005, Q5 MS

Q5 At the outset it should be emphasised that a large, well annotated diagram will enable insight into this question.

The first result may be obtained expeditiously by observing that if S_1 touches the sides BC, CA, AB at P, Q, R, respectively, then $AQ = AR = r \Rightarrow BR = c - r$, $CQ = b - r$. Thus $b - r + c - r = a \Rightarrow 2r = b + c - a$.

This result leads to $r = a(q-1)/2$ which is the key to the remainder of the question. In fact $R = [2bc - \pi a^2(q-1)^2]/\pi a^2$ (*).

From the data $a^2 = b^2 + c^2 \Rightarrow 2bc = (a+b+c)(b+c-a) \Rightarrow bc/a^2 = (q^2-1)/2$ which together with (*) leads to the second displayed result.

The obtaining of the turning value of a quadratic function is routine. In this context the method of completion of the square is to be preferred to the use of the calculus. Where the critical value of q is obtained from $dR/dq = 0$, it is important to give a reason as to why this defines an upper bound for R.