STEP II, 2005, Q3 MS

Q3 Here $dy/dx = x \sin x$ which is zero at $x = 0$ and is positive for $0 < x \leq \pi/2$. A further differentiation will show that $d^2y/dx^2 = 0$ at $x = 0$ and positive for $0 < x \leq \pi/2$. Also, as $y(0) = 0$ and $y(\pi/2) = 1$, then $0 \leq y \leq 1$ for $0 \leq x \leq \pi/2$, and the sketch can now be completed consistently with the above conclusions.

(i) $\int_0^{\pi/2} \sin x \, dx = \ldots = 1$ and use of the integration by parts rule will show that $\int_0^{\pi/2} x \cos x \, dx = \pi/2 - 1$. The displayed result for $\int_0^{\pi/2} y \, dx$ then follows immediately.

(ii) Start with $\int_0^{\pi/2} y^2 \, dx = \int_0^{\pi/2} \sin^2 x \, dx - \int_0^{\pi/2} x \sin 2x \, dx + \int_0^{\pi/2} x^2 \cos^2 x \, dx$.

Next, express $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$ so that now there are only two essentially different integrals involving trigonometric terms, namely, $\int_0^{\pi/2} x \sin 2x \, dx$ and $\int_0^{\pi/2} x^2 \cos 2x \, dx$. The second of these can be obtained from the first, again by application of the integration by parts rule. A correct application of this rule to the first integral and the careful collection of terms will lead to the displayed result.

For the final result, begin with the observation that $y^2 < y$ for $0 < x < \pi/2$. From this, it is immediate that $\int_0^{\pi/2} y^2 \, dx < \int_0^{\pi/2} y \, dx$ and hence that $\pi^3/48 - \pi/8 < 2 - \pi/2$. The final displayed result can then be obtained without difficulty.